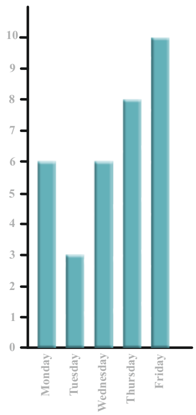
CUSTOMISED TEACHER'S TRAINING PACKAGE

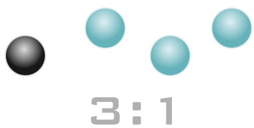
FOR KGBV TEACHERS

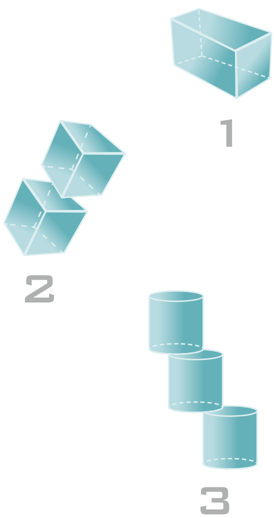
MATHEMATICS

TEACHING OF NUMBERS

BOOK 1







...Ahimsa is the very definition of woman and there is no place for untruth in her heart. If she is true to herself she is no longer Abala – *the weak, but she is* Sabala – *the strong...*

Customised Teacher's Training Package for KGBV Teachers (Mathematics)

Book 1

Teaching of Numbers

विद्यया ऽ मृतमश्नुते

एन सी ई आर टी
NCERT

DEPARTMENT OF WOMEN'S STUDIES

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्

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Foreword

National Curriculum Framework–2005 states that a critical function of education for equality is to enable all learners to claim their rights as well as to contribute to society and the polity. We need to recognise that rights and choices in themselves cannot be exercised until central human capabilities are fulfilled. Thus, in order to make it possible for all learners from different socio-economic backgrounds, especially girls, to claim their rights as well as play an active role in shaping collective life, education must empower them to overcome the disadvantages of unequal socialisation and enable them to develop their capabilities of becoming equal citizens.

Reaching out to the girl child has been central to the efforts of Universalising Elementary Education (UEE). The Sarva Shiksha Abhiyan (SSA), a national flagship programme for UEE recognises the need for special efforts to bring girls, especially from disadvantaged groups, to schools, and to bridge gender disparities in education at the elementary level. In this regard the Ministry of Human Resource Development instituted the Kasturba Gandhi Balika Vidyalaya (KGBV) scheme, an innovative and promising initiative that attempts to address the social, cultural and economic deprivation faced by girls from deprived and disadvantaged sections of rural society. Introduced as a scheme in 2004, it became a part of SSA in 2007. Currently it is operational in twenty-four States and one Union Territory.

A National Consultation on KGBV was organised by NCERT from 11–12 August 2008 to share experiences generated by the KGBV scheme over the last few years. This consultation brought together scholars in the field. The consultation strongly recommended development of Bridge Course for girls entering in KGBV and Customised Teacher

Training Package for upgrading the skills of KGBV teachers. Under this backdrop, Department of Women's Studies took initiative for developing Bridge Course and Teacher Training Package based on NCF-2005, in collaboration with other Curricular Departments of NIE, RIEs, University Departments, DIETs of Delhi, NGOs and practicing school teachers including teachers of KGBV. This material has been developed in the content areas of Science, Mathematics, History, Geography, Social and Political Life, Languages-English, Hindi and Art Education and is based on NCERT textbooks at elementary level.

The success of this effort largely depends on the multiple contextual steps that KGBV school principals and teachers would undertake to encourage girls to reflect on their own learning and to pursue imaginative activities and questions. We must recognise that, given space, time and freedom, girls generate new knowledge by engaging with the information passed onto them by adults. Treating the prescribed textbooks as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat girls as participants in learning and not as mere receivers of a fixed body of knowledge. The teacher should encourage girls to build on their own acquired and perceived knowledge and link it with their lived realities.

The present training material attempts to upgrade the skills of teachers during their in-service training in their subject areas of Science, Mathematics, History, Geography, Social and Political Life, Languages—English, Hindi and Art Education. Different participatory pedagogical methods have been adopted in all the subject areas to encourage activity-based teaching and learning. This material developed by NCERT can be treated as an initial material that can be supplemented later. It is not exhaustive in nature and it can be adopted or adapted according to the contextual needs of KGBV teachers.

The Department of Women's Studies (DWS) could not have gone ahead with this endeavour without the direction and guidance of Professor Krishna Kumar, former Director

NCERT. He had rightly envisioned the importance of the present Teacher Training Package in meeting the academic challenges of teachers of KGBV scheme.

We also gratefully acknowledge contributions of the Review Committee chaired by Dr. Sharda Jain, Director, Sandhan, Jaipur; and other members, Sister Sabina, former State Project Director, Mahila Samakhya Society, Patna, Bihar; Ms. Seema Bhaskaran, State Project Director, Mahila Samakhya Society, Kerala; Ms. Amukta Mahapatra, Director School Scape, Chennai for their expert review and suggestions. We are thankful to the members of Evaluation Team constituted by MHRD Ms. Sarita Mittal, Director EE8; Ms. Kiran Dogra, Consultant Gender, Ed. CIL; and Ms. Dipta Bhog, Director, Nirantar for their inputs and suggestions.

As an organisation committed to systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

Director
National Council of Educational
Research and Training

New Delhi
September, 2011

Preface

The training materials for the Kasturba Gandhi Balika Vidyalaya (KGBV) teachers have been developed keeping in view the principles of the National Curriculum Framework–2005 of the NCERT. These materials developed in different subject areas viz. English, Hindi, History, Geography, Social and Political Life, Arts Education, Science, and Mathematics are based on the NCERT upper primary textbooks. All these areas will contribute to the upgradation of professional skills of the KGBV teachers. These materials provide ample avenue to the KGBV teachers for their growth in pedagogy, methodology and approach in dealing with their subject areas. There is a considerable scope for exploration and creativity in the classroom. The use of bilingual technique in English will take teachers ahead in their thinking skills. The flexibility in the approach and suggested activities taking the help of worksheets, teacher demonstration, anecdotes, reciting poems, crossword puzzles, experimenting, hands-on skills, oral traditions and reading material across various subjects are the highlights of the manual.

Each subject area has picked up key concepts across the upper primary textbooks. Each concept has been dealt through a different kind of activity without bringing any definition and the content for rote learning. The concept or the idea has been floated through activities for the learners to catch and analyse. It is hoped that this material will be of use as a resource and also as reference material. The activities are suggestive. Any alternate activity can also be carried out based on the local-specific contexts. Each activity has the scope of creating similar other local-specific activities not making it necessary to stick to the materials given in this package. Its scope will get enhanced if this creates a space for more such activities.

The motivating material on Legendary Women of Science makes the training package of Science even more interesting and gives an edge for making it very gender sensitive. The women of India and the world, who have achieved heights in this area, will always encourage girls of the KGBVs to even explore these areas which have a masculine image. Kalpana Chawla will motivate girls to reach the heights of space while Florence Nightingale, the Lady with the Lamp, will encourage them to think in numbers. Marie Curie, the only woman to receive the Nobel Prize twice, will make them feel proud for being the women themselves. The KGBV teachers thus oriented to take up such challenges will certainly become guides and agents of social change.

In Mathematics there has been a conscious effort of demystifying the masculine image of mathematics. The processes underlying everyday mathematics done by women both within the home and outside have been highlighted. In several concepts, there is emphasis on the use of mathematic kit by teachers to make the learning of mathematics more concrete and useful for girls.

Women have always been the backbone of several historical and contemporary movements. Recognising their important role in social reform and national movement, the teacher training package in history highlights the contributions made by women like Rani Gaidinliu, Pandita Ramabai, Sarojini Naidu, Aruna Asif Ali and many more. Their trials and successes will continue to inspire girls to meet multifaceted challenges in life. Similarly the motivating material on legendary women like Rani Jhansi and Ila Sachani has been included in bridge course in languages. The training package for social and political life attempts to make teachers sensitive towards unconventional roles and responsibilities. Examples like Laxmi Lakra and Fatima Bi will touch hearts of common people. Geography equips teachers in spatial phenomenon. It instills human values and appreciations for regional inter-dependence and resource conservation. Arts and Aesthetics attempts to inspire diversity in expression of art. It is through performing art that all girls through various art forms can become communicative, creative and expressive.

Overall, keeping these variations in mind, the pedagogical approaches needed in the KGBVs will be multilevel and diverse for meeting the needs of KGBVs in different socio-cultural contexts.

New Delhi
September, 2011

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Contents

Teaching of Numbers

The ever need of the mankind was counting objects to count his/her belongings. This invented the idea of numbers, which in present system are termed as counting numbers or natural numbers.

"The numbers may be said to rule the whole world of quantity and the four rules of arithmetic may be regarded as the complete equipment of the mathematician."

– James Clerk Maxwell

Number sense is a critical component of Mathematics education. Encouraging students to estimate and check answers is an integral part of any numerical exercise. Discussing common mathematical situations with them and asking them to justify their mathematical conclusions will help students develop this crucial ability. Teaching of numbers should encourage number sense in the learners. Children with number sense pay attention to the meaning of numbers, operations on numbers and make realistic estimates of the results of computation. These abilities to deal sensibly with numbers are strength, both in and out of the classroom.

The system of natural numbers was gradually extended to other systems such as whole numbers, integers, rational numbers (including fractions and decimals) and real numbers as per need of the society and the subject. This block contains the following eight units:

Unit 1 Knowing Large Numbers
Unit 2 Natural Numbers and Whole Numbers
Unit 3 Integers
Unit 4 Fractions and Decimals
Unit 5 Rational Numbers
Unit 6 Squares and Square Roots
Unit 7 Cubes and Cube Root
Unit 8 Powers and Exponents

Unit 1 starts with the consolidation of the knowledge related to numbers up to five digits such as place value, comparison, expanded form and operations on them. Then, they have been extended to numbers having more than five digits. Estimation of numbers, writing of numbers using Roman numerals and use of brackets have also been discussed in this unit.

Unit 2 deals with natural numbers and whole numbers alongwith the properties of their operations. It also deals with factors and multiples, prime and composite numbers, even and odd numbers and tests of divisibility by some numbers. HCF and LCM of two or more numbers by factorisation alongwith their applications have also been discussed in this unit.

Unit 3 starts with the need of integers and deals with integers and operations on them alongwith their properties.

Unit 4 is mainly devoted to fractions, their classification such as like and unlike fractions, proper, improper and mixed fractions and their comparison including equivalent fractions. The processes of four fundamental operations on fractions have also been discussed in details in this unit. Decimals as special fractions with denominators 10, 100... have also been discussed in brief in this unit.

Unit 5 starts with the need of rational numbers and deals with rational numbers and operations on them alongwith their properties. The process of finding rational numbers between any two rational numbers has also been discussed in this unit.

Unit 6 deals with the process of finding squares of numbers and square roots of square numbers by factorisation. Finding square roots by division method has also been discussed in this unit, but it has been limited to the square roots, perfect square numbers or perfect square decimals only.

Unit 7 deals with finding the cubes of numbers and cube roots of some numbers by factorisation method. A trial and error method for finding the cube roots of a perfect cube (limited to a six digit number) has also been discussed in this unit.

Unit 8 is devoted to Powers and Exponents which begins with the representation of repeated multiplication as power or exponential notation. It discusses various laws of exponents. Writing of large numbers and small numbers in standard form has also been discussed in this unit.

Unit 1

Knowing Large Numbers

Structure

- Introduction
- Main Concepts/Sub-concepts
- Objectives
- Teaching Points
    1. Place value chart
    2. Expanded form of a number
    3. Comparison of numbers
    4. Ascending and descending orders
    5. Forming greatest and smallest numbers
    6. Word problems on different operations
    7. Estimation and rounding off
    8. Estimating outcomes of basic operations
    9. Use of brackets
    10. Roman numerals
    11. Extending numbers to higher places
- Common Errors
- Exercise

Introduction

In the primary bridge course classes, students have learnt about numbers up to 5 digits and have acquired an idea of place values of different digits in a given number. They had enough practice of reading and representing numbers on an abacus or in a place value chart. They are familiar with comparison of numbers and arranging them in ascending or descending order. Here, it is imperative to review some of these concepts to consolidate this knowledge and acquire mastery on the processes related to the above concepts. In order to understand applications of numbers and their operations, students must be encouraged to solve some word problems from daily life, based on fundamental operations involving large numbers conversion from larger to smaller units such as m to cm, kg to g and so on. Estimation is a very important skill to be developed in students. For this, it is necessary to first illustrate the process of approximation or rounding off numbers up to certain number of places say to the nearest ten or hundred or thousand and so on and then use it in estimating the outcomes of basic operations on them. Students may be given a feel by extending the place value chart that there can be larger and larger numbers.

In addition to the above, students may be exposed to the use of brackets and Roman numerals.

Main Concepts/Sub-concepts

Place value chart, reading and writing large numbers, Expanded form of numbers, Comparison of numbers, Ascending and descending orders, Word problems involving 5-digit numbers, Rounding off numbers, Estimation of the outcomes of different operations, Use of brackets, Roman numerals.

Objectives

After teaching the unit, students will be able to

- read and write numbers up to 5 digits, using place value chart;

- write numbers up to 5 digits in expanded form;
- compare numbers up to 5 digits;
- arrange given numbers (up to 5 digits) in ascending (or descending) order;
- form the greatest (or smallest) number using given digits;
- solve word problems based on different operations on numbers (upto 5 digits), including conversion from bigger unit to smaller unit, such as m to cm, kg to g, and so on;
- round off given numbers (upto 5 digits) to a specified place;
- estimate the outcomes of different operations on numbers (up to 5 digits);
- use brackets in different situations; and
- read and write Roman numerals upto 100.

Teaching Points

1. PLACE VALUE CHART

Teacher may ask the students to recall the reading and writing of numbers, through some examples, using the following place value chart:

Place Value Chart

T Th(10000)	Th (1000)	H (100)	T (10)	O (1)
5	8	3	8	7
4	1	0	3	3
	1	5	8	6
8	0	1	0	7

Students may be encouraged to write these numbers in the following way:

58387 Fifty eight thousand three hundred eighty seven

41033 ________ thousand __________hundred ________

1586 ________ thousand __________hundred ________

80107 ________ thousand __________hundred ________

Writing a number in figures is called **notation** and writing a number in words is called **numeration**.

Special emphasis may be given by the teacher in reading and writing numbers like 41033 or 80107, where the digit '0' is involved.

Initially, this process may be followed using the place value chart, but gradually student may be asked to do the same without the place value chart.

Teacher should make students clear about difference between face value, place, place value.

That is, face value → as digit itself

Place → Where the digit has been placed

Place value = face value × place

After enough practice, students may be apprised with the reverse process, i.e. writing the numbers given in words in figures as shown below:

Eighty one thousand two hundred ten = 81210

Twenty three thousand three hundred seven = 23307

Seventy thousand seventy six = _______

Two thousand four hundred six = _______

Ninety three thousand two hundred three = _______

Forty four thousand eighty nine = _______

2. EXPANDED FORM OF A NUMBER

Expanded form of

$42789 = 4 \times 10000 + 2 \times 1000 + 7 \times 100 + 8 \times 10 + 9 \times 1$

$65707 = 6 \times 10000 + 5 \times 1000 + 7 \times 100 + 0 \times 10 + 7 \times 1$

$40218 = 4 \times 10000 + 0 \times 1000 + 2 \times 100 + 1 \times 10 + 8 \times 1$

After explaining the above process, students may attempt the following questions:

$76554 = \underline{\quad} \times 10000 + \underline{\quad} \times 1000 + \underline{\quad} \times 100 + \underline{\quad} \times 10 + \underline{\quad} \times 1$

$45105 = \underline{\quad} \times 10000 + \underline{\quad} \times 1000 + 1 \times \underline{\quad} + \underline{\quad} \times 10 + \underline{\quad} \times 1$

$54720 = \underline{\quad} \times \underline{\qquad} + \underline{\quad} \times 1000 + \underline{\quad} \times 100 + 2 \times 10 + \underline{\quad} \times 1$

$80309 = \underline{\quad} \times 10000 + \underline{\quad} \times 1000 + \underline{\quad} \times 100 + 0 \times 10 + \underline{\quad} \times 1$

Here again more emphasis may be laid on the numbers involving the digit '0'.

3. COMPARISON OF NUMBERS

Students are familiar with the process of comparing two numbers. After having discussion on comparison of numbers through concrete examples as given in bridge course, teacher may ask them to recall the following rules for comparing two numbers, through appropriate examples:

Rule 1: Of the two numbers, the one having more number of digits is greater.

For example, 24135 > 5439 and 5623 > 897. While writing in the above form, students may be acquainted with the symbols > (meaning greater than) or < (meaning less than). Then, they may be told that the above relations can also be written as

$5439 < 24135$ and $897 < 5623$, respectively.

Rule 2: If the two numbers have the same number of digits, then the numbers are compared on the basis of their leftmost digits, i.e. the number with greater leftmost digit is greater.

For example, $32406 > 12587$,

$47613 > 38598$, etc.

In case, the leftmost digits are the same, then the next left most digits are compared.

For example, $65324 > 63876$,

$47207 > 45899$, etc.

In case, these two leftmost digits are equal, then their next leftmost digits are compared and the process is continued until you come across unequal digit at corresponding places. Clearly, the number with greater such digit is the greater of the two.

For example, $65218 > 65127$,

$65782 > 65753$, etc.

Note: If the teacher feels that the students are not aware of the above two rules, then she must first devote some time on the concrete examples based on comparison of numbers as given in bridge course of mathematics and then gradually evolve these abstract rules of comparison by taking examples from daily life involving small numbers (such as 250 mangoes and 187 mangoes, 27 km and 23 km, etc.) and then take up the large numbers for comparison.

4. ASCENDING AND DESCENDING ORDERS

Teacher should first explain the meaning of arranging numbers in ascending (or descending) order by taking some smaller numbers. For this, an **activity** of arranging students of the class **according to their heights** may be performed. Once the meaning of ascending or descending order is clear, then the larger numbers up to 5 digits may be taken up and using the Rules 1 and 2 given above, students may be encouraged to arrange numbers in ascending (or descending) order themselves. Teacher should intervene only when some mistakes are made by them.

Students may also be made aware with the writing of numbers in the above orders using the symbols > and <.

Example: Arrange the numbers 47891, 8798 and 56652 in ascending order.

Solution: The required arrangement is

8798, 47891, 56652

Using the symbol <, this can be written as 8798 < 47891 < 56652.

Example: Arrange the numbers 9897, 58765 and 49781 in descending order.

Solution: The required arrangement is

58765, 49781, 9897

Using the symbol >, this can be written as 58765 > 49781 >9897

5. FORMING GREATEST AND SMALLEST NUMBERS

As an **activity**, ask students to form different numbers with given three digits and let them decide which number is the greatest and which number is the smallest. For example, with the digits 2, 1 and 8, the numbers formed will be

128, 182, 218, 281, 812, 821

The **greatest** of these numbers is **821** and the **smallest** of these numbers is **128**.

Using the digits 5, 0 and 3, the 3 digit numbers formed will be

503, 530, 305, 350

By taking more such examples make the students observe that in the greatest number, i.e. 821, the given digits 1, 2 and 8 are occuring in descending order of magnitude and in the smallest number, i.e. 128 the given digits 1, 2 and 8 are occuring in ascending order.

Here, it may be pointed out to the students that 053 and 035 have not been considered as the desired numbers, because 0 in the leftmost place has no meaning. In such a case, it becomes a two digit number.

The **greatest** of these numbers is 530 and the **smallest** of these numbers is 305.

Give enough practice to the students for forming greatest and smallest 3 digit numbers with given three digits and then gradually extend this idea to numbers with four and five digits.

Example: Form the smallest and the greatest four digit numbers, using the digits without repetition:

(i) 7, 5, 6 and 4

(ii) 2, 0, 5 and 9

(iii) 8, 3, 2 and 3

Solution: (i) Smallest number : 4567

Greatest number : 7654

(ii) Smallest number : 2059

Greatest number : 9520

(iii) Smallest number : 2338

Greatest number : 8332

Example: Form the smallest and the greatest five digit numbers, using the digits without repetition:

(i) 8, 5, 6, 4 and 9

(ii) 7, 5, 0, 3 and 6

(iii) 6, 3, 2, 5 and 2

Solution: (i) Smallest number : 45689

Greatest number : 98654

(ii) Smallest number : 30567

Greatest number : 76530

(iii) Smallest number : 22356

Greatest number : 65322

Teacher may help students to generalise that forming a **greatest** 5 digit number is simply arranging the digits in ascending order and to form a **smallest** 5 digit number, digits are to be arranged in desending order.

6. WORD PROBLEMS ON DIFFERENT OPERATIONS

Before taking up word problems, ensure that students have fully understood the four fundamental operations on numbers. It may begin with simple word problems as given below and gradually complex word problems may be solved:

Example: For a National Relief Fund, students of School A collected Rs 22325, that of School B collected Rs 15225 and that of School C collected Rs 25370. Find the total collection of the three schools.

Solution: Total collection (in Rs) is:

$$\begin{array}{r} 22325 \\ 15225 \\ +\ \underline{25370} \\ \underline{62920} \end{array}$$

Thus, total collection is Rs 62920.

Example: Anjali opened her Savings Bank Account with a sum of Rs 25000. After sometime, she withdrew Rs 5450 from it. How much money was left in her account?

Solution: Money left = Rs 25000 – Rs 5450

$$\begin{array}{r} \text{Rs}\ \ 25000 \\ -\ \underline{\text{Rs}\ \ \ \ 5450} \\ \underline{\text{Rs}\ \ 19550} \end{array}$$

Thus, money left in Anjali's account is Rs 19550

Example: A basket contains 56 apples. A fruit merchant buys 45 such baskets. How many apples did he buy?

Solution: Apples in one basket = 56

Therefore, apples in 45 baskets = 56 × 45 = 2520

Example: 332 saplings are to be planted in rows containing 13 saplings in each row. How many more saplings will be needed to complete each row?

Solution: To find the number of rows, 332 shall be divided by 13 as shown below:

```
13 ) 332 ( 25 ——— Quotient
     26
    ----
      72
      65
    ----
       7 ——— Remainder
```

Here, it may be seen that 25 rows are completely filled with saplings, but 7 saplings are there in the 26th row. So, number of saplings needed to complete the 26th row = 13 – 7 = 6

Example: Nalini purchased 2 kg 350 g apples, 3 kg 800 g mangoes and 1 kg 450 g bananas from a shop. What quantity of fruits did she purchase?

Solution: Here, it may be emphasised that all the weights should be first converted into smaller units.

2 kg 350 g = 2350 g (In this case it is gram)

3 kg 800 g = 3800 g

+ 1 kg 450 g = 1450 g (As 1 kg = 1000 g)

Total weight = 7600 g

Thus, total fruits purchased by Nalini is 7600 g, i.e. 7 kg 600 g.

Example: The cloth required to stitch a suit is of length 3 m 25 cm. Find the length of the cloth required to stitch 6 suits.

Solution: Cloth required to stitch one suit = 3 m 25 cm = 325 cm

Therefore, cloth required to stitch 6 suits

= 325 cm × 6

= 1950 cm

Thus, required length of the cloth is 1950 cm, i.e. 19 m 50 cm

After explaining the solutions of these problems, teacher may encourage students to solve some more problems of these types from their day-to-day life.

7. ESTIMATION AND ROUNDING OFF

Estimation means to make a guess. Through this, we get a rough or approximate idea of the answer to a question involving operations on numbers.

Through examples it may be emphasised to the students that estimation is a very useful process in day-to-day life.

Some examples of estimation are as follows:

1. The amount of flour, rice, pulses, vegetables, etc. taken by your mother or father to prepare meals.
2. Prior arrangement of money for constructing a house.
3. Keeping enough quantities of different items for sale in a shop on the basis of their demand and supply.
4. Making annual budget provisions for different sectors such as education, health, defence, sports, etc.
5. Making arrangements for different celebrations or events such as marriage parties, exhibitions, school functions, etc.

To explain the idea further, students attention may be drawn to the statement such as '76500 people live in a particular city'. Here, it must be brought to the notice of the students that it is difficult to keep track on how many people live in a city at a particular time. They might be a little more than 76500 or a little less than 76500. They might not be exactly 76500. Thus, 76500 is most probably an **estimated number**. Students may be apprised with the fact that there are various techniques of estimation and one of these techniques is based on rounding off numbers to nearest ten, hundred and so on. After this, explain the process of rounding off numbers through some examples as given below

Example: Round off 5237 to the nearest
(i) ten (ii) hundred (iii) thousand

Solution:
(i) Students may be asked to remember that **rounding off** a number to the **nearest ten** means to round it off to the **nearest multiple of ten.** It may be observed that 5237 lies between 5230 and 5240. Now, 5237 – 5230 = 7 and 5240 – 5237 = 3, i.e. 5237 is closer to 5240 than to 5230. Thus, 5237 is rounded off to the nearest ten as 5240.

(ii) Here again students may be asked to remember that rounding off a number to the nearest hundred means to round it off to the **nearest multiple** of **hundred**.

It may also be observed that 5237 lies between 5200 and 5300.

It may also be observed that 5237 is **more closer to 5200** than to 5300, because 5237 – 5200 < 5300 – 5237.

Thus, 5237 **rounded off** to the **nearest hundred** is 5200.

(iii) Here, it may be observed that 5237 lies between 5000 and 6000. It may also be observed that 5237 is more closer to 5000 than to 6000, because 5237 – 5000 < 6000 – 5237

Thus, 5237 rounded off to the **nearest thousand** is 5000.

Example: Round off 5750 to the nearest hundred.

Solution: Ask the students to proceed in the same manner as in the previous example. That is, it may be observed that 5750 lies between

5700 and 5800. Also, it may be observed that 5750 – 5700 = 50 and 5800 – 5750 = 50.

Thus, 5750 is equally closer to both 5700 and 5800.

At this stage, the **convention** that in such a case, 5750 is **rounded off upwards** to 5800 (and not to 5700) may be explained to the students. The teacher may give some more examples of the above two types and on the basis of these examples, state the following two rules for rounding off numbers to the nearest ten (or hundred or thousand or upwards):

Rule 1: If the digit at the one's place is **less** than 5, then the number is rounded off **downward** by keeping ten's place digit as it is and one's place digit is replaced by the digit 0 for rounding off to the **nearest ten**. Similarly, if the digit at the ten's place is less than 5, then the number is rounded off to the **nearest hundred** by keeping the hundred's place digit as it is and each of the digits at ten's place and one's place is replaced by 0 and the same process is continued for onward places.

Rule 2: If the digit at the **one's place is equal to or greater than** 5, then the number is rounded off to the **nearest ten** by **increasing** the value of the digit at the ten's place **by 1** and replacing the one's digit by 0. Similarly, if the digit at the ten's place is **equal to or greater than** 5, then the number is rounded off to the **nearest hundred** by **increasing** the value of the **digit** at the hundred's place **by 1** and replacing each of the ten's and one's place digits by 0. The same process is continued for upward places.

While applying the above rules, special attention may be made in the **following type of rounding off involving the digit '9'.**

Example: Round off (i) 86972 to the nearest hundred (ii) 79568 to the nearest thousand.

Solution: (i) In 86972, digit at the ten's place is 7 (i.e. greater than 5). Therefore, for rounding off this number to the nearest hundred, digit at the hundred's place (i.e. 9) is to be increased by 1 and each of the digits at ten's and one's places are to be replaced by 0. Now, 9 increased by 1 will become 10, therefore the digit at the hundred's place will become 0 and 1 will be carried to the thousand's place to make 6 as 7.

Thus, 86972 rounded off to the **nearest hundred** is 87000.

(ii) In 79568, digit at the hundred's place is 5. Therefore, for rounding off the number to the nearest thousand, the digit at the thousand's place (i.e. 9) is to be increased by 1 and each of the digits at hundred's, ten's and one's places are to be replaced by 0.

Thus, 79568 rounded off to the **nearest thousand** is 80000.

8. ESTIMATING OUTCOMES OF BASIC OPERATIONS

After ensuring that the students have acquired mastery on rounding off numbers, they may be explained the process of estimation of outcomes of basic operations through the following types of examples. Teacher should make children familiar with the need of using operation and estimation together. For example, she can tell students, if they have to go for shopping and they carry a list to the shop, they need to visualise before leaving home the money they should carry? (This gives the idea about estimation).

Example: Estimate the outcome of 59 + 83 after rounding off these numbers.

Solution: Rounded off to the nearest ten, 59 = 60 and rounded off to the nearest ten, 83 = 80

Therefore, 59 + 83 can be estimated in the form 60 + 80 = 140

Thus, estimation of the outcome of 59 + 83 is 140.

Example: Estimate the outcome of 6345 – 1724 after rounding off these numbers.

Solution: Rounded off to the nearest thousand, 6345 as 6000 and rounded off to the nearest thousand 1724 as 2000.

Therefore, 6345 – 1724 can be estimated as 6000 – 2000 = 4000.

Thus, estimated outcome is 4000.

Note : It may be noted that 6345 and 1724 can be rounded off to the **nearest hundred** as 6300 and 1700, respectively. Then, 6345 – 1724 can be **estimated** as 6300 – 1700 = 4600.

Thus, one can have different estimated answers depending on the place of rounding off.

Example: Estimate 4753 × 43 after rounding off these numbers.

Solution: Rounding 4753 to the nearest hundred, you will get 4800 and rounding off 43 to the nearest ten, you will get 40. Therefore, 4753 × 43 will be estimated as 4800 × 40 = 192000.

Note: It may be noted that 4753 could have also been rounded off to the nearest ten (instead of nearest hundred) as 4750. In that case 4753 × 43 would have been estimated as 4750 × 40 = 190000.

Example: Estimate 722 ÷ 63 after rounding off these numbers.

Solution: To the nearest ten, 722 is 720 and to the nearest ten 63 is 60.

Therefore, 722 ÷ 63 should be estimated as 720 ÷ 60 = 12.

Through these examples, it may be brought home before the students that different estimations may be obtained for a single outcome of an operation, depending upon the place or places the numbers involved, are rounded off. It may also be emphasised in the class, that there is no hard and fast rule for obtaining quick estimate for the outcome of an operation, but sometimes the following **general rule** can give a **quick estimation**.

'Round off each of the numbers involved in the operation to its highest place and then perform the actual operation.'

9. Use of Brackets

Use of brackets may be explained to the students by giving them certain situations, where two or more terms of an expression are considered as a single group. For example, to divide 35 by the sum of 2 and 5, '2 + 5' is considered as a single group and written as 35 ÷ (2 + 5). Simplification of the brackets should also be explained through suitable examples as given in the Mathematics textbook.

10. Roman Numerals

One of the earliest systems of writing numerals is through Roman Numerals.

While introducing these numerals, it may be told to the students that symbols used to denote numbers are called **numerals**. The numerals that we are using are known as **Hindu-Arabic numerals**. It may also be told that different symbols (i.e. numerals) had been used by different civilizations. Further, Roman numerals are still being in use at a number of places (say on the clock). There are seven basic symbols to write any numeral. These are I V X L C D M

Then, the students may be made familiar with the following symbols used in Roman System of Numeration:

I	II	III	IV	V	VI	VII	VIII	IX	X
1	2	3	4	5	6	7	8	9	10
L		C	D		M				
50		100	500		1000				

Following rules regarding Roman numerals may also be explained through examples:

1. If a symbol is repeated, its value is added as many times it repeats.
2. A symbol is not repeated more than three times. V, L and D are never repeated.
3. If a symbol of smaller value is written to the right of a symbol of greater value, its value gets added to the symbol of the greater value.
4. If a symbol of smaller value is written to the left of a symbol of greater value, its value is subtracted from the greater value.
5. Symbols V, L, D and M are never written to the left of symbol of greater value, i.e. V, L, D and M are never subtracted.
6. I can be subtracted from V and X only, X can be subtracted from L, M and C can be subtracted from D and M only.

Use of these numerals may be explained through various examples and counter examples.

Example: Write 23, 39 and 82 in Roman numerals.

Solution: 23 = XXIII

39 = XXXIX

82 = LXXXII

Example: Write XC, LVI and LXXIV in Hindu Arabic Numerals

Solution: XC = 100 – 10 = 90

LVI = 50 + 5 + 1 = 56

LXXIV = 50 + 10 + 10 + 5 – 1 = 74

Example: What are the meanings of XIIII, XXII, VVI and DC?

Solution: XIIII and VVI have no meaning.

XXII = 22 and DC = 600

Here, it may be pointed out that in Roman numerals 39 = XXXIX; it should not be counted that X has repeated '**four times**'. Repetition should be counted when the symbols are placed adjacent to each other. Enough drill work should be assigned to the students to acquire mastery on the concepts.

11. Extending Numbers to Higher Places

Students may be exposed to the extension of numbers to higher places by considering largest two digit number (99), largest three digit number (999), largest four digit number (9999), and so on and adding 1 to each of them as follows :

99 + 1 = 100 (one hundred)

999 + 1 = 1000 (one thousand)

9999 + 1 = 10000 (ten thousand)

99999 + 1 = 100000 (one lakh)

999999 + 1 = 1000000 (ten lakh)

9999999 + 1 = 10000000 (one crore) and so on.

Then with the help of following place value chart, reading and writing of such numbers may be explained

Crore	Ten L	L (Lakh)	T Th	Th	H	T	O
10000000	1000000	100000	10000	1000	100	10	1

Common Errors

(i) Place value

(a) Place value of 2 in the number 45269 is 100

(b) Place value of 0 in the number 570394 is 1000

(ii) Reading/Writing of Numbers

(a) Ten thousand two hundred ten: 102010

(b) 20220: Two thousand two hundred twenty

(iii) Comparison of numbers

(a) 3953 < 499 because 39 < 49

(b) 10010 < 9999

(iv) Writing number in descending/ascending order. Sometimes, students use these terms one for another, i.e. for descending order, they write ascending order and vice versa.

(v) In word problems, conversion from bigger to smaller units, like kg to g. They take 100 g = 1 kg. Similarly, 1 km = 100 m.

(vi) Forget to write 0 in the quotient in between in division questions. For example.

$$1701717 \div 17 = 111$$

$$18071 \div 17 = 163$$

(vii) Forget to multiply the number 0 if it appears as a digit in a number. For example.

$$19205 \times 23 = 44275$$

$$\begin{array}{r} 19205 \\ \underline{\times\ 23} \\ 5775 \\ \underline{3850\times} \\ \underline{44275} \end{array}$$

(viii) Rounding off numbers

(a) 2050 is rounding off to the nearest hundred as 2000 instead of 2100

(b) 39728 is rounded off to the nearest thousand as 310000

(ix) Writing numbers in Roman numerals

4 → IIII in place of IV

95 → VC in place of XCV

You may evaluate students through the following exercise.

Exercise

1. Find the greatest and the smallest numbers in 25286, 25274 , 25245 , 25210 and 26002.
2. Arrange 55450, 85400, 84500, 7861 and 7725 in ascending and descending orders.
3. Write the place value of 7 in 857489.
4. Write 89324 in the expanded form
5. Write 783029 in words.
6. In an election, the successful candidate got 577589 votes and the nearest rival got 348708 votes. By what margin did the candidate win the election ?
7. A vessel has 4 litres 500mL of curd. In how many glasses, each of 25mL capacity, can this curd be filled?
8. Estimate the following by rounding off:

   (i) 17986 + 5290 (ii) 5673 × 436
9. Find the value of (10 + 7) × 109 by simplifying the brackets.
10. Write in Roman numerals:

    (i) 72 (ii) 94
11. Write in Hindu Arabic numerals:

    (i) XXVI (ii) XCV

Unit 2

Natural Numbers and Whole Numbers

Structure

- Introduction
- Main Concepts/Sub-concepts
- Objectives
- Teaching Points
    1. Natural numbers
    2. Whole numbers
    3. Operations on whole numbers
    4. Some number patterns
    5. Factors and multiples
    6. Prime and composite numbers
    7. Even and odd numbers
    8. Tests of divisibility
    9. Common factors and common multiples
    10. Prime factorisation
    11. HCF and LCM
- Common Errors
- Exercise

Introduction

We all know that the study of mathematics has begun with human experiences with nature. The first ever need of the human beings was the counting of their belongings or objects. This fundamental need had evolved the numbers 1, 2, 3, 4... called **counting numbers** or **natural numbers**. In fact, whatever numbers had been discussed so far were natural numbers. These numbers were helpful in counting different objects such as cattles, persons in a building and so on. However, the problem arised like 'how to indicate, there is no person in the room'. This problem was solved by including the number 0 (zero). The natural numbers along with 0 are called **whole numbers**. As the students have already performed different fundamental operations on numbers, therefore in this unit only properties of the operations on natural numbers and whole numbers are being discussed. In addition to the above, concepts of factors and multiples along with tests of divisibility by some numbers are also being considered. Based on factors and multiples, finding of HCF and LCM of number along with their applications are also to be discussed.

Main Concepts/Sub-concepts

Natural numbers, Predecessor and successor of a number, Whole numbers, The number line, Properties of different operations on whole numbers, Some patterns in numbers, Factors and multiples, Prime and composite numbers, Co-prime, even and odd numbers, Tests of divisibility by 2, 3, 4, 5, 6, 8, 9, 10 and 11, HCF and LCM by factorisation.

Objectives

After teaching of this unit, students will be able to:

- distinguish between natural numbers and whole numbers;
- write the predecessor and successor of a given number;

- state the smallest natural number and the smallest whole number;
- add, subtract or multiply smaller numbers, using the number line;
- state the various properties of different operations on whole numbers such as closure property, commutative property, associative property, identity element, distributive property, etc. and use them in solving problems;
- identify certain number patterns and use them in solving problems;
- write the factors and multiples of given numbers;
- identify prime and composite numbers in a given collection;
- understand that 1 is neither prime nor composite;
- identify pairs of co-prime numbers from a given collection;
- identify even and odd numbers from a given collection;
- state the tests of divisibility of numbers by 2, 3, 4, 5, 6, 8, 9, 10 and 11 and use them in solving problems;
- write the common factors of two or more numbers;
- find the HCF of two or more numbers by factorisation;
- write the common multiples of two or more numbers; and
- find the LCM of two or more numbers by factorisation.

Teaching Points

1. NATURAL NUMBERS

By now, students have worked with numbers in different situations. They may be told that all these numbers are called natural numbers, because it appears that these numbers have come to the mind of the human being in a natural way. Thus, 1, 2, 3, 4, ... are **natural numbers**.

It may be pointed out that the number 2 is obtained by adding 1 to 1, number 3 is obtained by adding 1 to 2, and number 4 is obtained by adding 1 to 3 and so on.

In view of the above, 2 is called the **successor** of 1, 3 is called the **successor** of 2, 4 is called the **successor** of 3 and so on.

As a reverse of this process, 1 is called the **predecessor** of 2, 2 is called the **predecessor** of 3, 3 is called the **predecessor** of 4 and so on. From this type of discussion, it may be brought home to the students that 1 has **no predecessor** in natural numbers and hence 1 is the **smallest natural number**.

Now, you may ask the student to write the largest natural number. After getting that number, you may add 1 to that number and explain that the number given by her is not the largest natural number and there is no end to this process. This will help the students to understand that there is **no largest natural number.**

2. WHOLE NUMBERS

As already stated, if a collection of numbers including 0 and natural numbers is formed then we get the numbers 0, 1, 2, 3, 4,... called **whole numbers**. By following the same procedure as in the case of natural numbers, you can illustrate that 1 is the successor of 0, 2 is the successor of 1 and so on. Similarly, 0 is the predecessor of 1, 1 is the predecessor of 2 and so on. Further, 0 has no predecessor in whole numbers and hence it is the **smallest whole number**.

Also, by following the same activity as done for natural numbers, it may be brought home to the students that there is **no largest whole number**.

At this stage, the teacher may ask the students to solve some questions of Exercise 2.1 of Class VI, Mathematics, NCERT.

During the discussion on whole numbers, it is important to note that 0 has a double role. In one role, it acts as a **number (zero)** to represent no object and in the other role it acts as a **digit (0)**, i.e. it acts as a **place holder** in the place value system.

The number line: To discover some more facts about the whole numbers, they can be represented on a line called the **number line**. In this, whole numbers are marked at equal distances as shown in the figure given below:

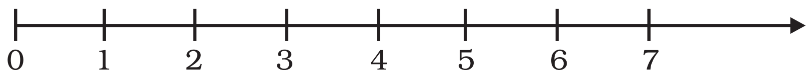
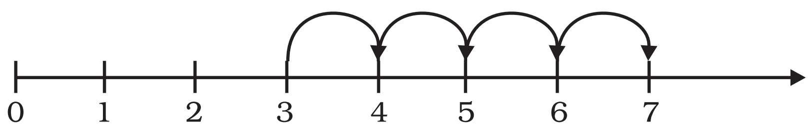


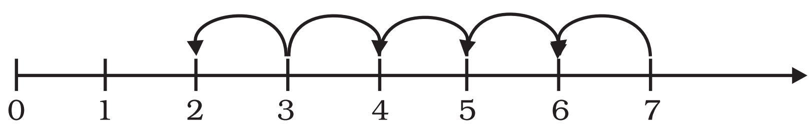
Fig. 2.1

It can be observed that on the number line, 7 lies to the right of 3 and also it is known that 7 > 3. Similarly, 8 lies to the right of 5 and 8 > 5. Thus, it can be observed that the number which lies to the right of a number on the number line is greater than the other. Similarly, a number which lies to the left of a number on the number line is less than the other number. Addition, Subtraction and Multiplication of whole numbers are shown in Fig. 2.2, Fig. 2.3 and Fig. 2.4, respectively.



Representing 3 + 4 = 7

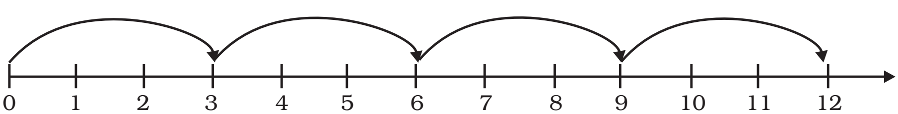
Fig. 2.2



Representing 7 – 5 = 2

Fig. 2.3

Students may be asked to note the directions of the arrows carefully in Fig. 2.2 (addition) and in Fig. 2.3 (subtraction).



Representing 4 × 3 = 12

Fig. 2.4

The teacher may explain these ideas by taking more examples based on number line, because they will be more useful in the study of other collections of numbers such as integers, rational numbers, etc.

3. Operations on Whole Numbers

Students are already familiar with the four fundamental operations on whole numbers. Here, the need is to highlight some of the properties of these operations. These can be illustrated through some examples as given below:

a. Addition

(i) **Whole numbers are closed under addition, i.e. sum of two whole numbers is a whole number**

For example 9 + 4 = 13, 8 + 9 = 17, 9 + 0 = 9 and so on.

(ii) **Addition of whole numbers is commutative**

For example 6 + 9 = 9 + 6, 273 + 185 = 185 + 273, 89 + 125 = 125 + 89, and so on.

It may be pointed out that due to this property, **two whole numbers can be added in any order.**

(iii) **Addition of whole numbers is associative**

For example, (5 + 9) + 6 = 5 + (9 + 6), (82 + 123) + 16 = 82 + (123 + 16), (75 + 9) + 421 = 75 + (9 + 421), and so on.

It may be pointed out that due to this associative property, one can write the sum of three whole numbers say (5 + 9) + 6 or 5 + (9 +6) as simply 5 + 9 + 6. Similarly, (82 + 123) + 16 or 82 + (123 + 16) can simply be written as 82 + 123 + 16 and so on.

It may also be emphasised that by repeatedly applying the commutative and associative **properties of addition, it is possible to add any number of whole numbers in any order**.

(iv) **Identity element for addition or additive identity:** 0 is the additive identity for **whole numbers**, because 5 + 0 = 5 = 0 + 5, 23 + 0 = 23 = 0 + 23, and so on.

b. Subtraction

The operation of subtraction is an inverse process of addition.

(i) Whole numbers are **not closed** for subtraction, because 5 – 3 = 2 is a whole number, but 4 – 8 is not a whole number.

(ii) Subtraction is not commutative for whole numbers, because $5 - 3 \neq 3 - 5$, $12 - 11 \neq 11 - 12$, etc.

(iii) Subtraction of whole numbers is not associative. For example, $(18 - 12) - 4 \neq 18 - (12 - 4)$, etc.

(iv) 0 is not an identity element for subtraction. For example, 6 – 0 = 6 but $0 - 6 \neq 6$; 23 – 0 = 23, but $0 - 23 \neq 23$, etc.

c. Multiplication

(i) **Whole numbers are closed under multiplication,** i.e. product of two whole numbers is a whole number.

For example, 8 × 9 = 72, 47 × 12 = 564, 87 × 0 = 0, and so on.

(ii) **Multiplication of whole numbers is commutative**

For example, 6 × 9 = 9 × 6, 87 × 125 = 125 × 87, and so on. It may be pointed out that due to this property, **two whole numbers can be multiplied in any order.**

(iii) **Multiplication of whole numbers is associative**

For example, (9 × 5) × 3 = 9 × (5 × 3), (17 × 12) ×115 = 17 × (12 × 115), and so on. It may be pointed out that due to this property, one can write the product of three whole numbers say (9 × 5) × 3 or 9 × (5 × 3) as simply 9 × 5 × 3 . Similarly, (17 × 12) × 115 or 17 × (12 × 115) can be simply written as 17 × 12 ×115, and so on.

It may also be emphasised that by repeatedly applying the commutative and associative properties of multiplication, one **can multiply any number of whole numbers in any order**.

(iv) **Identity element for multiplication or multiplicative identity for whole numbers**

1 is the multiplicative identity for whole numbers, because 7 × 1 = 7 = 1 × 7, 95 × 1 = 95 = 1 × 95, and so on.

d. Division

Division is the inverse operation of multiplication

(i) Whole numbers are not closed under division, because 8 4 = 2 is a whole number, but 5 ÷ 15 is not a whole number, etc.

(ii) Division is not commutative for whole numbers, because 16 4 4 16, 20 ÷ 5 5 20, and so on.

(iii) Division of whole numbers is not associative, because (120 20) ÷ 2 120 (20 ÷ 2), (75 15) ÷ 5 75 (15 ÷ 5) and so on.

(iv) 1 is not an identity element for division of whole numbers because 8 ÷ 1 = 8, but 1 8 8; 29 1 = 29, but 1 ÷ 29 29.

During the discussion of division of whole numbers, it may be repeatedly explained that division by zero is not allowed (or not defined). For this, the activity of repeatedly subtracting 0 from a number may be performed in the class, as given on pages 33 and 34 of the Class VI, Mathematics, NCERT.

(e) Distributive property of multiplication over addition

This property is very important and is useful in the further study of Mathematics. It may be explained through the activity suggested on page 38 of Class VI, Mathematics, NCERT (Cutting of a graph sheet in two parts and then counting the number of squares in each part).

The property may also be explained by considering four small numbers such as 9, 8 , 5 and 3 and showing that 9 × 8, i.e. 9 × (5 + 3) = 9 × 5 + 9 × 3.

By actual multiplication, 9 × (5 + 3) = 72 and 9 × 5 + 9 × 3 = 45 + 27 = 72 .

Thus, 9 × (5 + 3) = 9 × 5 + 9 × 3.

Similarly, it can be shown that 27 × (11+ 8) = 27 × 11 + 27 × 8, and so on.

It may be emphasised **that multiplication is distributive over subtraction also**.

Thus, 27 × (11 – 8) = 27 × 11 – 27 × 8, 35 × (29 – 12) = 35 × 29 – 35 × 12 , and so on.

After explaining the above properties, students may be asked to solve questions of Exercise 2.2 of Class VI textbook by **using suitable properties or suitable rearrangement of numbers**. As a teacher, you may act as a facilitator during whole process of problem solving.

4. SOME NUMBER PATTERNS

Mathematics is known as science of patterns. In fact in about the attributes of things, children apply reasoning to answer 'what is next?' not with a number but with a description. Basically, it may always be a nice tool to initiate mind action in an interesting way. Mathematics is especially useful when it helps you to predict and number patterns are all about prediction input. Recognising number patterns is also an important problem-solving skill. Many of the number patterns have been given on pages 41 to 43 of Class VI, Mathematics, NCERT. You may collect some more patterns and ask the students to complete them.

For example

(i) $1 + 3 = 4 = 2 \times 2$

$1 + 3 + 5 = 9 = 3 \times 3$

$1 + 3 + 5 + 7 = 16 = 4 \times 4$

$1 + 3 + 5 + 7 + 9 =$ ___ = ___ × ___.

(ii) $2 + 4 = 6 = 2 \times 3$

$2 + 4 + 6 = 12 = 3 \times 4$

$2 + 4 + 6 + 8 = 20 = 4 \times 5$

$2 + 4 + 6 + 8 + 10 =$ ___ = ___ × ___ and so on.

Students may also be encouraged to make their own number patterns while solving questions on page 42 – 44 of Class VI, Mathematics, NCERT.

5. FACTORS AND MULTIPLES

Concepts of 'factors' and 'multiples' can be explained to the students by asking them to find the product of some given numbers. For example, it will be seen that

$5 \times 8 = 40$ (1)

$9 \times 3 \times 6 = 162$ (2)

$3 \times 7 \times 11 = 231$ (3)

and so on.

From the above examples, it may be illustrated that factors and multiples are very closely related terms in the following manner:

In (1), 5 and 8 are the **factors** of 40 and 40 is a **multiple** of both 5 and 8.

In (2), 9, 3 and 6 are **factors** of 162 and 162 is a **multiple** of each of the numbers 9, 3 and 6.

In (3), 3, 7 and 11 are the **factors of 231 and 231 is** a **multiple** of each of the numbers 3, 7 and 11.

If a number '*a*' divides a number '*b*' exactly then '*a*' is called a factor of '*b*' and '*b*' is called a multiple of '*a*'.

Let the students play the Game 1 'Fractors of Numbers' in the Mathematics Kit given with this package. Make the students more confident about this concept through this game.

Example

Factors of 12 are 1, 2, 3, 4, 6 and 12, because 12 is divisible by all these numbers. Similarly, factors of 91 are 1,7,13 and 91, factors of 144 are 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72 and 144, and so on.

Multiples of 2 are 2, 4, 6, 8, 10, ...

Multiples of 5 are 5, 10, 15, 20, 25, ...

Multiples of 12 are 12, 24, 36, 48, 60, 72, and so on.

Now students may be asked to attempt questions of Exercise 3.1, page no 50 of Class VI Textbook. While solving these questions, following features of factors and multiples may be highlighted:

1. 1 is a factor of every number.
2. Every number is a factor of itself.
3. Every factor of a number is equal to or less than the number.
4. Number of factors of a given number is finite.
5. Every multiple of a given number is greater than or equal to that number.
6. Every number is a multiple of 1 and itself.
7. Number of multiples of a given number is infinite.

Here, it may also be mentioned that **factors** are also called **divisors.** It may also be pointed out that, in general, unless stated otherwise, we talk of **factors** and **multiples** only in the case of **natural numbers**.

6. PRIME AND COMPOSITE NUMBERS

You may ask the students to note down the number of factors of some numbers and record their observations in the following form:

Numbers having only one factor	Numbers having only two factors (1 and the number itself)	Numbers having more than two factors
1	2,3,5,7,........	4,6,8,9,10, 12,14,16, ...

You may ask the students to prepare a table as given on page 51 of Class VI, Mathematics, NCERT and then define the prime and composite numbers as follows:

Numbers having only two factors (1 and the number itself) are called **prime numbers,** e.g. 2, 3, 5,7,11...

Numbers having more than two factors are called **composite numbers,** e.g. 4,6,8,9,10,12....... **Number 1** is **neither prime** nor **composite.**

Now the students may be exposed to the **Sieve of Eratosthenes for the listing of prime number** provided by a Greek mathematician of third century B.C. It is given on page 52 of Class VI, Mathematics, NCERT.

7. Even and Odd Numbers

Students may be exposed to the numbers 2,4,6,8. Ask them to divide each of these numbers by 2. They might say that remainder in each case is 0. Thus, all these numbers are multiples of 2. Then, a formal definition of even numbers can be given as follows:

A number divisible by 2 is called an **even number**. Thus, 2,4,6,8,10, ... are even numbers. The rest of the numbers 1,3,5,7....... are called **odd numbers.**

Here, again it must be emphasised that, unless stated otherwise, we talk of even and odd numbers only in the case of natural numbers. After discussing even and odd numbers, these may be linked with prime numbers by stating the following two properties:

(i) **2 is the smallest prime number and it is even** In fact, **it is the only prime number which is even.**

(ii) Every prime number, except 2 , is an odd number

Now, the students may be asked to attempt questions of Exercise 3.2 given on pages 53 and 54 of Class VI, Mathematics, NCERT.

Here, it may be mentioned that answers to many of these questions will not be unique. For example, in question 10(b), 31 = 5 + 7 + 19 and 31 = 7 + 11 + 13, etc.

8. TESTS OF DIVISIBILITY

To find whether a given number is divisible by another number, we have to perform actual division and check the remainder to get the answer. By using divisibility test of a number we can know the answer by simply examining the digits of the given number.

Teacher may start discussion with the questions like 'find whether the number 122, 231 are divisible by 2 or not?' and gradually by increasing the number of digits, she may continue the discussion, to emphasise the need of some divisibility rules to make the entire process easy. Students may be asked for divisibility by 2, 3, 4, 5, 6, 8, 9, 10 and 11 one by one. Based on their observations, following divisibility tests may be stated as given on page 54 to 56 Class VI, Mathematics, NCERT:

Divisibility by 2: A number is divisible by 2, if it has any of the digits 0,2,4,6 or 8 in its ones place. For example, 1358 is divisible by 2 but 2257 is **not** divisible by 2.

Divisibility by 3: A number is divisible by 3, if the sum of its digits is divisible by 3. For example, 261 is divisible by 3, because 2 + 6 + 1 = 9, which is divisible by 3. But 265 is not divisible by 3, because 2 + 6 + 5 = 13 is not divisible by 3.

Divisibility by 4: A number (with 3 or more digits) is divisible by 4 if **the number formed by its last two digits (i.e, tens and ones digits) is divisible by 4.** For example, 9732 is divisible by 4, because number formed by its last two digits, i.e, 32 is divisible by 4. However 286 is not divisible by 4, because 86 is not divisible by 4.

It may be pointed that in case of 2700, the number formed by last two digits is 00. In such cases also (i.e. when the number of zeroes are in pairs), the number is divisible by 4. If the last digits of a number is zero then it may not be divisible by 4.

Divisibility by 5: A number is divisible by 5, if the digit at its ones place is either 0 or 5. For example, 3295 and 42390 are divisible by 5, but 3756 is not divisible by 5.

Divisibility by 6: A number is divisible by 6, if it is divisible by both 2 and 3. For example, 450 is divisible by 6, because it is divisible by both 2 and 3. But 453 is not divisible by 6, because it is divisible by 3 but not by 2. Similarly, 448 is not divisible by 6, because it is divisible by 2 but not divisible by 3.

At this point, you may pose a question before the students like

12 is divisible by 2 and 12 is divisible by 4. Is it true that 12 is divisible by 2 × 4?

Answer to this question is 'No'. Here, you may explain that a number can only be divisible by the product of its divisors when the **two divisors have no common factors.** In 2 and 4, there is a common factor 2 while in the case of 2 and 3 , there is **no common factor other than 1.**

Divisibility by 8: A number of four or more digits is divisible by 8, if the number formed by its last three digits, i.e. at hundreds, tens and ones places is divisible by 8. For example, 2104 is divisible by 8, because 104 is divisible by 8 , but 2220 is not divisible by 8 because 220 is not divisible by 8.

Also, as into the case of divisibility by 4, if the number 224000 is considered, then the number formed by its last three digits is '000'. On actual division 224000 is divisible by 8. Therefore, in such cases, you may say that if a number ends in three zeroes, then it is divisible by 8. For example 21000 is divisible by 8, but 21500 is not divisible by 8.

Divisibility by 9: A number is divisible by 9, if the sum of its digits is divisible by 9. For example, 4608 is divisible by

9, because 4 + 6 + 0 + 8 = 18 is divisible by 9. However 6518 is not divisible by 9, because 6 + 5 + 1 + 8 = 20 is not divisible by 9.

Divisibility by 10: A number is divisible by 10, if the digit at its one's place is 0. For example, 4870 is divisible by 10, but 8005 is not divisible by 10.

Divisibility by 11: If the difference between the sum of the digits at odd places (from the right) and the sum of the digits at even places (from the right) of a number is either 0 or divisible by 11, then the number is divisible by 11. For example, 61809 is divisible by 11, because (9 + 8 + 6) – (0 + 1) = 23 – 1 = 22 is divisible by 11.

↑ odd places   ↑ even places

Also, 1331 is divisible by 11, because

(1 + 3) – (3 + 1) = 0

↑ odd places   ↑ even places

But 2663 is not divisible by 11, because (3 + 6) – (6 + 2) = 1 is not divisible by 11.

After stating these tests, it may be mentioned that if a number is divisible by 10, then it is also divisible by its factors namely 5 and 2.

Similarly, if a number is divisible by 9, then it is also divisible by its factor 3. If a number is divisible by 8, then it is also divisible by its factors 4 and 2. **But it may be noted that the converses of the above statements are not true.**

Teacher may summarise the divisibility tests as follows:

Order : 2, 5, 10, 3, 9, 6, 4, 8, 11

This may help the children to understand the similarity between the rules

e.g.	Rule	Numbers
→	Last digit	2, 5, 10
→	Sum of digits	3, 9
→	Rule of 6 to be given only after rule of 2 and 3	
→	Last 2 digits	4
→	Last 3 digits	8
→	Rule of 11	

After illustrating these tests, students may be asked to attempt questions of Exercise 3.3 (page 57) of Class VI, Mathematics Textbook, NCERT.

The following two properties may also be illustrated through examples:

(i) **If the two given numbers are divisible by a number, then their sum is also divisible by that number.** For example, 14 is divisible by 2 and 18 is divisible by 2. Therefore, (14 + 18), i.e. 32 is also divisible by 2.

(ii) **If the two given numbers are divisible by a number, then their difference will also be divisible by that number.**
For example, 36 is divisible by 6 and 24 is divisible by 6. Therefore (36 – 24), i.e. 12 is divisible by 6.

9. Common Factors and Common Multiples

For understanding the concepts of common factors and common multiples, teacher may encourage students for discussion on real life situations to emphasise the need of them. The students may be asked to list factors and multiples of two or more numbers and to select common factors and common multiples from these lists as given on pages 58 and 59 of Class VI, Mathematics, NCERT. Special care must be taken, while finding common factors or common multiples of more than two numbers.

The common factor of two or more numbers must be in all the numbers. For example, take the factors of 18, 45 and 12.

They are as follows:

$$18 = 3 \times 3 \times 2$$

$$45 = 3 \times 3 \times 5$$

$$12 = 2 \times 2 \times 3$$

In the above, it will not be correct to say that the common factor of 18, 45 and 12 is 2, because it is a common factor in 18 and 12 only and not present in all the numbers 18, 45 and 12.

Similarly, 3×3, i.e. 9 is not a common factor of 18, 45 and 12, because it is common only for 18 and 45 and **not** in all the numbers 18, 45 and 12.

The same condition holds for common multiples of more than two numbers. After this classification, students may be asked to attempt questions of Exercise 3.4 of page 59 of Class VI, Mathematics, NCERT. At this stage, it may be explained that **if in two numbers, there is no common factor other than 1, then the two numbers are said to be co-prime.**

For example, 5 and 12 are co-prime, 3 and 13 are co-prime, 9 and 16 are co-prime, and so on.

It may also be explained that in the pair of co-primes, **both the numbers may be primes.** For example (3 and 13), **one number prime and the other composite** (5 and 12) **or both the numbers composite** (9 and 16).

Here, it may also be stated that **if a number is divisible by a pair of co-prime, then that number is divisible by the product of those co-prime.** In this regard, students may be asked to recall the test of divisibility of a number by 6, which is a product of co-prime 2 and 3.

10. PRIME FACTORISATION

Expressing a number in the form of a product of its prime factors only is called the **prime factorisation** of the number. For example, $2 \times 2 \times 3$ is the prime factorisation of the number

12, but 4 × 3 is not a prime factorisation of 12, because 4 is not a prime number.

1 × 2 × 2 × 3 is also not a prime factorisation of 12, because 1 is not a prime number.

It may also be noted that every composite number can be factorised into primes in only one way except for the order of primes.

This concept may also be explained using **'Tree Diagrams'** as given on pages 60 and 61 of Class VI Textbook. After this, students may attempt Exercise 3.5 of Class VI Textbook (Page nos. 61 and 62).

11. HCF AND LCM

HCF and LCM should be discussed in the continuation of the concept of **common factors** and **common multiples.** From the list of common factors of two or more numbers, greatest or highest of the common factors will be called the **highest common factor** (HCF) of the numbers. Sometimes, it is called the greatest common divisor (GCD). Similarly, from the list of common multiples of two or more numbers, the smallest or least of the common multiples will be called the **least common multiple (LCM) of the numbers.** Sometimes, it is also referred to as **lowest common multiple**.

HCF (p,q) × LCM (p,q) = p×q, but this is valid in case of 2 numbers only.

After explaining these concepts, through common factors and common multiples, HCF and LCM of two or more numbers may be obtained through the process of prime factorisation as explained on page nos. 63, 64 and 65 of Class VI Mathematics Textbook.

After this, students may be asked to attempt questions of Exercise 3.6 of Class VI Mathematics Textbook (page no. 63) and Exercise 3.7 (pages 65 and 66) of Class VI Mathematics Textbook. However, before asking the students to attempt Exercise 3.7, they may be first asked to attempt the following questions:

1. Find the HCF of the following numbers by prime factorisation method :

(i) 48, 60
(ii) 12, 15, 45
(iii) 18, 77
(iv) 6, 15, 18, 30
(v) 45, 105, 165
(vi) 420, 240, 660
(vii) 9, 10, 11
(viii) 25, 35, 45
(ix) 33, 34
(x) 28, 35, 77

2. Find the LCM of following numbers by common division method :

(i) 88, 216
(ii) 480, 600
(iii) 21, 35, 42
(iv) 69, 115, 253
(v) 108, 135, 162
(vi) 15, 30, 90
(vii) 180, 384, 144
(viii) 12, 168, 266
(ix) 21, 42, 14, 28
(x) 450, 600, 900, 180

While solving the question like finding HCF of 18 and 77 by prime factorisation, you may write that 18 = 2 x 3 x 3 and 77 = 7 x 11

In these prime factors, there is **no common factor.** Some students may write that HCF of 18 and 77 is 0, because **there is no common prime factor.** They may be explained that in such a case HCF is 1 and not 0, because 1 is a factor of all the numbers.

After solving the above type of questions, students may be asked to understand carefully examples 12, 13 and 14 (Pages 65 and 66) of Class VI Mathematics Textbook. After this, they may attempt questions of Exercise 3.7 (page 67) of Class VI Mathematics Textbook.

During this discussion of HCF and LCM, following properties may also be highlighted

(i) HCF of two or more numbers is smaller than or equal to the smallest of the numbers.

(ii) LCM of two or more numbers is greater than or equal to the largest of the numbers.

(iii) HCF of two numbers is a factor of their LCM and LCM of two numbers is a multiple of their HCF.

(iv) If the HCF of two numbers is one of the numbers, then their LCM is the other number.

(v) HCF of co-prime numbers is 1 and LCM of co-prime numbers is their product.

Common Errors

(i) Identity element in Whole Numbers

0 is taken as identity element in subtraction also by considering 4 – 0 = 4, 8 – 0 = 8, 11 – 0 = 11, but 0 – 4 $\neq$ 0, 0 – 8 $\neq$ 0 and 0 – 11 $\neq$ 11

(ii) Reciprocal of 0 is taken as , which is not defined.

(iii) Difference of numbers 243 and 341 is taken as 243 – 341 = – 98 which is a whole number.

(iv) In the collection of natural numbers, predecessor of 1 is taken as 0, which is a whole number.

(v) In whole numbers, predecessor of 0 is taken as – 1, which is an integer.

(vi) $\frac{1}{0} = 0$, $\frac{1}{0} = 1$.

(vii) 0 × 1 = 0, 0 × 2 = 0, therefore 0 × 1 = 0 × 2 or 1 = 2.

(viii) Factor and multiple of a number are used one for another.

(ix) 1 is the prime natural number.

(x) 0 is an even natural number.

(xi) All odd natural numbers are primes.

(xii) Smallest even number is 0.

(xiii) Smallest prime number is 1.

(xiv) Factors of 36 are 2, 3, 4, 6, 9, 12, and 18. Forget to write 1 and 36.

(xv) To examine the test of divisibility by a number say 7, students may use the divisibility test of 3 or 9, i.e. they add up the digits and see whether the sum is divisible by 7, which is not correct. However, divisibility by 7 is not given at this stage.

(xvi) If a number is divisible by 4 and 6, then it will be divisible by 4 × 6 = 24, which is not correct. They may be using the result that if a number is divisible by 2 and 3 then it is divisible by 6.

(xvii) While finding HCF of 9 and 11 by prime factorisation, they may say that there is no HCF of 9 and 11 or HCF is 0 as there is no common factor, but 1 is a factor of every number.

(xviii) Prime factorisation of 36 is 1 × 2 × 2 × 3 × 3

(xix) A number is divisible by 5, then it is also divisible by 10.

(xx) In solving word problems involving HCF and LCM, they find LCM instead of HCF and vice versa.

You may evaluate your students through the following exercise

Exercise

1. Write the predecessor of 10000.
2. Out of 370 and 307, which of the whole numbers is on the left of the other on the number line?
3. Find the sum 1962 + 453 + 1538 + 647 using a suitable rearrangement.
4. Write the next two steps of the following pattern: 1 × 8 + 1 = 9; 12 × 8 + 2 = 98; 123 × 8 + 3 = 987; ...
5. Give an example to show that division is not associative for whole numbers.
6. Write all the factors of 56.
7. Write first three multiples of 7.
8. Write the largest and smallest prime numbers between 1 to 100.

9. Determine whether 10824 is divisible by 11 or not.
10. Write the prime factorisation of 980.
11. Find the HCF of 12, 45 and 75.
12. Find the LCM of 20, 25 and 30.
13. Length, breadth and height of a room are 825 cm, 675 cm and 450 cm, respectively. Find the longest tape that can measure the dimensions of the room exactly.
14. The traffic lights at three different road crossings change after every 48 seconds, 72 seconds and 108 seconds, respectively. If they change simultaneously at 7 am; at what time will they change simultaneously again?
15. Find the least number which when divided by 16, 24 and 32, leave remainder 3 in each case.

Unit 3

Integers

Structure

- Introduction
- Main Concepts and Sub-concepts
- Objectives
- Teaching Points
    1. Need for negative numbers
    2. Introducing integers
    3. Representation of integers on a number line
    4. Ordering of integers
    5. Numerical (absolute) value of an integer
    6. Operations on integers
        - 6.1 Addition
        - 6.2 Subtraction
        - 6.3 Multiplication
        - 6.4 Division
    7. Properties of the operations on integers
- Common Errors
- Exercises

Introduction

Students know about natural numbers, i.e. 1,2,3,4,5,... and whole numbers, i.e., 0,1,2,3,4... They also know the use of these numbers in daily life. The children are now aware of the four basic operations on these numbers and also the properties of natural numbers and whole numbers on these operations, like commutative, associative and distributive. Here, we shall discuss about another system of numbers called integers.

Main Concepts and Sub-concepts

- Representation of integers on number line.
- Addition of integers.
- Additive inverse.
- Subtraction of integers.
- Properties of addition and subtraction of integers.
- Multiplication of integers.
- Properties of multiplication of integers.
- Division of integers.
- Properties of division of integers.
- Absolute value of an integer.

Objectives

After teaching this unit the students will be able to

- understand the meaning of an integer (negative, zero and positive integers);
- representing integers on the number line;
- identify integers from a given collection of numbers;
- write the numerical value of an integer;
- add integers;
- state the properties of addition of integers through examples;

- subtract integers;
- add and subtract large integers using properties of integers;
- multiply integers;
- state the properties of multiplication of integers through examples;
- multiply large integers using properties of integers;
- divide integers;
- use integers to describe different situations in daily life; and
- solve some daily life problems involving operations on integers.

Teaching Points

1. NEED FOR NEGATIVE NUMBERS

The teacher may give some situations and contexts which highlight the need to extend the system of whole numbers to a system of numbers which includes negative numbers also. One such situation is given below:

Situation

Ruchika and Salma are playing a game using a number strip as shown below:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Each of them places a token of different colour at 0. Two coloured dice, say red and blue are placed in a bag. When red coloured dice is taken out, the token has to be moved to the right, and for blue dice the token has to be moved to the left as per the numbers on the dice. The dice is put back into the bag after each move. The one who reaches the 25th mark first, is the winner. The game starts!

Ruchika gets a red dice. She throws it and gets number 4 on it. She moves her token to the mark 4. Salma also gets a red dice with 3 on it and puts her token at 3.If in her second attempt Ruchika gets 5 on the blue dice, where do you think she should put her token? When she starts counting numbers on the strip to the left of 4, she finds that she needs one more marking to the left of 0. By which number should it be represented? Shall we use the same numbers to the left, i.e. 1, 2, 3 ..., etc.? If we do that, will it not create confusion?

Such questions may be asked and discussion can be generated among students on it.

These situations will help the students to get a feel for the need of numbers other than those they have studied so far. Such numbers may be referred to as negative numbers.

Negative Numbers

In order to fulfill the need of more numbers we make use of the same symbols used for writing the numbers already studied along with a '–' sign. So we use symbols –1, –2, –3. Make use of the same strip in the situation given above. Extend the strip to the left of 0 as –1, –2, –3, ..., –25.

You may get such a strip prepared by the students and make them play the game played by Ruchika and Salma as in the above situation.

The child understands the concept quickly if it is presented in a visual manner. The numbers –1, –2, –3 ... may be referred to as negative numbers.

2. Introducing Integers

If negative numbers such as –1, –2, –3, ... are included in the system of whole numbers, we get the numbers ..., –3, –2, –1, 0,1,2,3, ... These numbers are called integers.

The numbers 1,2,3, ... are called positive integers whereas –1, –2, –3 ... are called negative integers. The number 0 is neither a positive nor a negative integer. The numbers 0,1,2,3, ... are also called non-negative integers.

Teacher may emphasise the difference of positive and non-negative integers.

Clearly –1 and +1 are at equal distances from 0 but in opposite directions. This will be true for all corresponding positive and negative integers. In our daily life we come across situations that are opposites of each other.

Example

(i) Height above the sea level is 500 km implies +500 km. But depth below the sea level is 500 km implies –500 km.

(ii) Gain of ₹ 100 implies +100 but loss of ₹ 100 implies –100.

(iii) Increase of 10 implies +10 but decrease of 10 implies –10.

Teacher may give many more such examples from real life situations to emphasise and make the concept of integers clear to the students.

You may ask the students as to what else can you call the collection of numbers (i) 1,2,3,... (ii) 0,1,2,3, ...

Of course

(i) is a collection of natural numbers, i.e. positive integers

(ii) is a collection of whole numbers, i.e. non-negative integers

The relation between these systems of numbers can also be represented by a diagram.

Let us understand this by the following figures. Let us suppose that the figures represent the collection of numbers written against them.

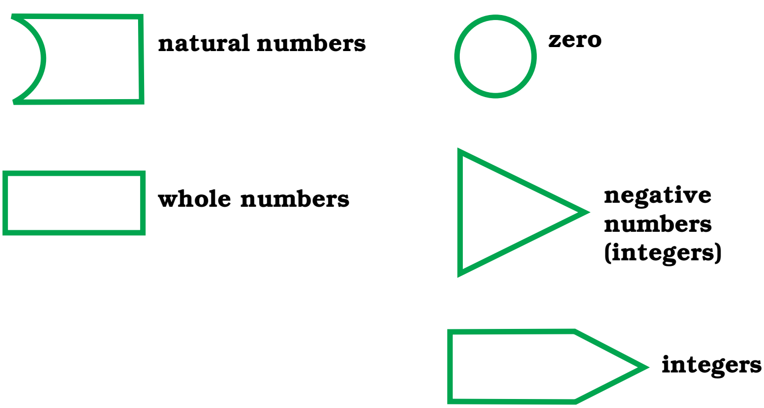


Fig. 3.1a

Then the collection of integers can be understood by the following diagram in which all the earlier collections are included:

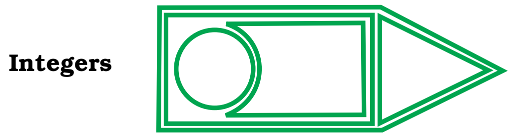


Fig. 3.1b

At this stage, a student may understand the concepts effectively if they are presented through a visual mode. We thus use a number line for this purpose.

3. Representation of Integers on a Number Line

The whole numbers are represented on a number line as shown below:

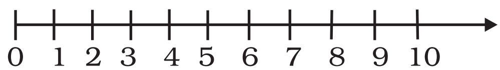


Fig. 3.2

For representing integers we extend this line to the left of 0 as shown below:

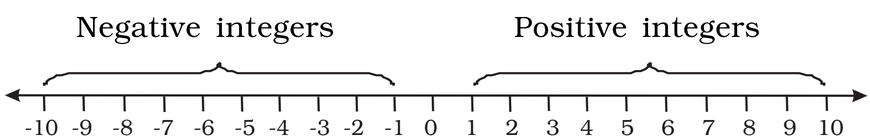


Fig. 3.3

Markings to the right of 0 represent positive integers and to the left of 0 represent negative integers.

The point –7 is 7 units to the left of 0, etc.

Teacher may ask the students to encircle some points on the number line representing different numbers. For example,

(i) Encircle –7, 5, –9, 7, etc.

(ii) Identify the point which is 6th on the left of 0, 7th on the right of 0, etc.

(iii) Identify which point is marked immediately, to the left of –6, right of –6, left of 1, right of 1, etc.

These exercises will help the students to get an idea of the placement of integers on the number line. This idea can further be strengthened by asking them questions on the number strip used by Salma and Ruchika in their game. For example,

(i) Salma's counter is at –2. She gets 4 on a blue dice. Where should she keep her counter now?

(ii) Ruchika's counter is at 2. She gets 5 on a red dice. Where should she keep her counter now?

Many such questions should be asked to the students. This activity firmly settles the arrangement of integers on the number line in the student's mind.

This also gives the students an idea of ordering of the integers.

4. Ordering of Integers

The idea of ordering the integers may be given through an activity, followed by questions on it. The activity discussed earlier helps us here. Take the number line shown below:

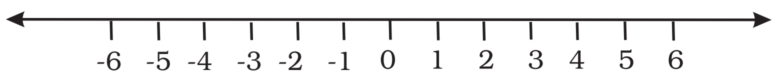


Fig. 3.4

Teacher may ask questions like

(i) Which number is immediately to the left of –7, right of –9, left of 0, right of 1, etc.?

(ii) Which number comes two units to the right of –8, three units to the left of 2, four units to the right of –3, etc.?

After this, take the help of ordering of positive integers (i.e. 1, 2, 3 ...), which the students have already learnt to give them an idea of ordering of negative integers.

A discussion may be initiated through following questions based on number line:

(i) 1 is to the right of 0, and 1 > 0. 2 is to the right of 1 and which is greater 2 or 1 ? Continue like this.

Do this for many such pairs of numbers.

(ii) (a) List five numbers that are greater than 7.

(b) To which side of 7 will they occur on the number line? (right or left)

(iii) (a) List five numbers that are less than 15.

(b) To which side of 15 will they occur? (right or left)

(iv) (a) Spot any five numbers to the right of 10 on the number line.

(b) Are they greater or less than 10?

Do this in respect of consecutive numbers also. Give enough practice of this kind.

The notion thus develops in the mind of the student that numbers increase towards the right and decrease towards the left on the number line.

Reinforce this idea through such questions as given below:

(i) Which are the integers smaller than –5? (List 10 such integers)

(ii) Which are the integers greater than –10? (List 10 such integers)

(iii) Write the integers between –8 and –1 ; –5 and 5, etc.

Thus, the integers are ordered as

$\ldots < -3 < -2 < -1 < 0 < 1 < 2 < 3 < \ldots$

Let the students do the following exercises:

1. Compare the following pairs of integers using > or <.

(i) 0 ☐ –5 (ii) –6 ☐ 7

(iii) 8 ☐ –8 (iv) –100 ☐ 1

(v) –7 ☐ –10

2. Write the integers on the markings given on the number line.

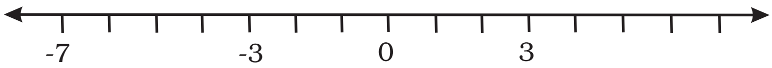


Fig. 3.5

5. NUMERICAL (ABSOLUTE) VALUE OF AN INTEGER

Consider the integer +7 on the number line. It is at a distance of 7 units to the right of 0. You will find that –7 is also at a distance of 7 units to the left of 0. Both the integers +7 and –7 are at the same distance of 7 units from 0 but are located in opposite directions. This number (or distance) 7 associated with both the integers +7 and –7 is called the numerical (absolute) value of +7 and –7.

We can thus say, that the absolute value of an integer is its numerical value regardless of its sign. For example,

Absolute value of – 2 is 2

Absolute value of + 4 is 4

6. OPERATIONS ON INTEGERS

Students have learnt the various operations of whole numbers on a number line. The same method can be extended for operations of integers using number line.

6.1 ADDITION

We know that there are positive integers, negative integers and 0. Thus, while adding integers we have to add.

(i) Two positive integers
(ii) Two negative integers
(iii) One positive and one negative integer.
(iv) 0 and any other integer.

This can be introduced through different visual ways.

(a) Through a daily life situation.
(b) Through an activity of mathematics kit
(c) Through the number line.

(a) Through a daily life situation

Lata is standing on the first floor of her school building. There is one more floor above the first floor. There are 10 steps each going towards second floor and ground floor.

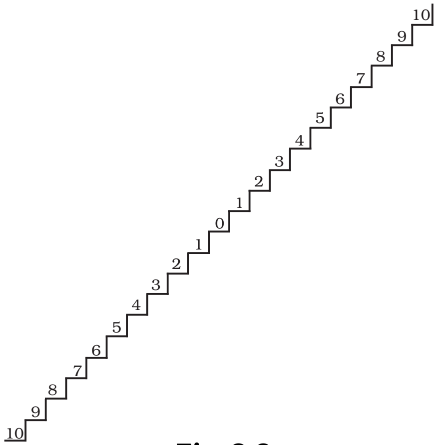


Fig. 3.6

We suppose that climbing a certain number of steps upwards is taken as a positive integer and going down as a negative integer. First floor is taken as 0 level.

(i) She climbs 4 steps up from 0 level and from there goes 2 steps further up. Where will she reach? In terms of numbers, we write it as 4+2 = 6. She is at 6th step above first floor.

(ii) She climbs 6 steps up from 0 level and from there comes down 3 steps. Counting the steps we get that she is at step 3 above first floor. That is 6 + (–3) = 3

(This example also helps in ordering of integers).

(iii) She goes down 5 steps from 0 level and from there goes down 3 steps. Counting the steps we find that she is 8 steps below the first floor. Numerically, (–5) + (–3) = –8.

(iv) She goes down 4 steps from 0 level and from there climbs up 3 steps. We find that she is 1 step below the first floor. Numerically, (–4) + 3 = –1

Give many such examples of different such combinations of positive and negative integers.

(b) Through an activity of mathematics kit

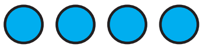
Activity 1 entitled 'Addition and Subtraction of Integers' given in mathematics kit, may be utilised for a concrete experience of this concept to start with. Alternatively, following activity could also be performed:

In case of question like: (–3) – (–4)

 : '–'

 : '+'

(–3) means 

(–4) means 

Subtraction means 'To take away', i.e. from three blue tokens (represented as –3) we have to take away 4 blue tokens (represented as –4).

Since we cannot take away four blue tokens from three blue tokens, so we may add a zero pair to three blue tokens as given below

Now we can easily take away four blue tokens, so the left out is  = +1, the required answer.

(c) Addition through number line

Here we adopt the convention that adding a positive integer will mean moving towards right and adding a negative integer will mean moving towards left on the number line.

(i) Let us see what 3 + 5 is equal to.

For doing this first locate 3 on the number line. Since a positive integer 5 is being added to 3 so we move further 5 steps to the right of 3. Where do we reach?

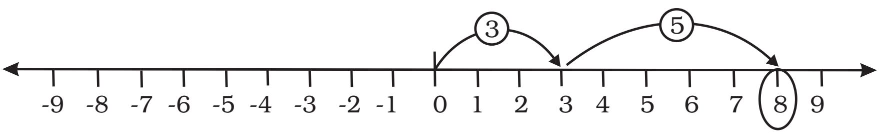


Fig. 3.7

We reach 8. So, 3+5=8 (Fig. 3.7). Thus, adding any positive integer means moving that many steps to the right of the number.

(ii) (–5) + 3

Locate –5 on the number line. Since a positive integer is being added, so move 3 steps to the right of –5.

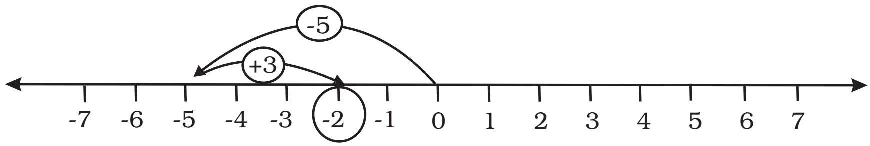


Fig. 3.8

We reach at –2 (Fig 3.8). Thus, adding any negative integer means moving that many steps to the left of the number, (–5) +3 = –2.

(iii) (–3) + (–5)

Locate –3. Since a negative integer –5 is being added so move 5 steps to the left from –3.

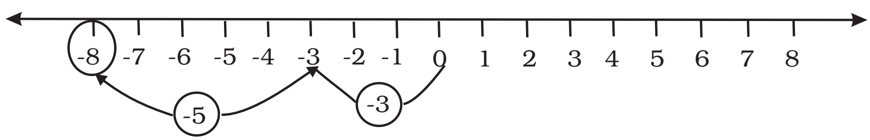


Fig. 3.9

We reach at –8 (Fig. 3.9.) So, (–3) + (–5) = –8

In this way, take many such combinations of integers and let the students understand the addition of integers in a playful manner.

Ultimately they should arrive at the notions that:

(i) addition of two positive integers is done as they have already learnt for natural numbers;

(ii) while adding two negative integers, addition should be done as done for positive integers, but the answer takes a minus sign. For example, (–2) + (–3)= – (2 + 3) = –5; and

(iii) when one positive and one negative integer are added, forget the signs and subtract the smaller number from the bigger one. Put the sign of the number having absolute value bigger number in the result so obtained.

So, while doing (–5) + 3, forget the signs of –5 and 3. We have two numbers 5 and 3, and 5 > 3. And, 5 – 3 = 2. The sign of number called bigger number i.e. 5 is '–'. So, (–5) + 3 = –2.

Similarly, for 4 + (–7), 7 – 4 = 3. So, 4 + (–7) = –3 [Since 7 > 4 and sign of 7 in the given addition is '–']

Please remember that we do not put the sign of '+' before a positive integer. We simply write +1 as 1, +2 as 2 and so on.

Ask the students to add 0 to any integer and let them find the sum as

8 + 0 = 0 + 8 = 8 , –7 + 0 = 0 + (–7) = –7 and so on.

Let them take many more such examples and observe that $a + 0 = 0 + a = a$, where a is any integer.

We say that 0 is the additive identity for integers.

Additive inverse

Let the students add 3 and –3 on the number line as shown in Fig. 3.10

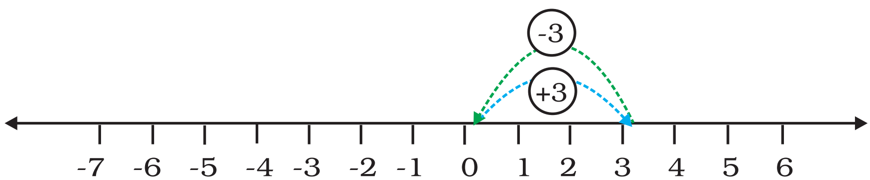


Fig. 3.10

From the above, 3 + (–3) = 0.

Similarly, (–5) + 5 = 0

We say that 3 and –3 are *additive inverses*, of each other, i.e. the additive inverse of 3 is –3 and that of –3 is 3. Similarly, 5 and –5 are *additive inverses* of each other.

A number added to its additive inverse gives zero.

Ask the students to do the following questions:

1. Using the number line, write the integer which is

   (i) 2 more than 7 (ii) 2 less than 5

   (iii) 3 more than –4 (iv) 3 less than –6

2. Add the following:

(i) 5 + (–7) (ii) –3 + (–8)

(iii) 11 + (–2) (iv) 2 + (–3) + (–7)

6.2 SUBTRACTION

To give an idea of the subtraction of two integers, again make use of the visual medium, i.e. the number line. Take different combinations of integers. Make use of the conventions used earlier of moving towards right or left on a number line while adding a positive or a negative integer, respectively with the difference that in case of subtraction we move in the opposite direction for the numbers to be subtracted.

(i) Take 6 – 2

From our earlier experience of subtraction of whole numbers, we know that, 6 – 2 = 4. On the number line also, we first locate 6, and to subtract 2, move 2 steps to the left from 6. We reach at 4. So, 6 – 2 = 4.

Fig. 3.11

The same movement on the number line can also be interpreted as 6 + (–2). We have seen this while adding two integers.

Thus, we can say, 6 – 2 is same as 6 + (–2) (Fig. 3.11).

(ii) 6 – (–2)

First we decide the direction of movement from 6. Whether it is 2 steps to the left or to the right of 6. We know that for 6 + (–2), we move 2 steps to the left of 6 to arrive at 4. Can we do the same for 6 – (–2)? No! We will move in the opposite direction, i.e. we move to the right of 6 by 2 steps and reach at 8 as shown in Figure 3.12.

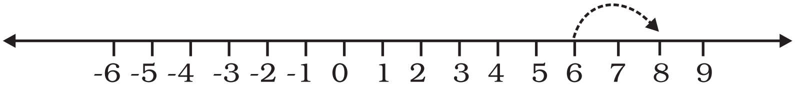


Fig. 3.12

Thus, 6 – (–2) =8.

This movement on the number line can also be seen as 6 + 2.

So, 6 – (–2) = 6 + 2.

Again we see that the additive inverse of –2 is 2. Therefore, 6 – (– 2) = 6 + (additive inverse of –2).

In a similar way, we get

(–5) – 2=(–5) + (additive inverse of 2) = (–5) + (–2) = –7 (as done earlier for addition)

Thus, by making use of number line and the concept of additive inverse of an integer, the concept of subtraction of two integers can be reduced to the concept of addition of two integers, which the students are already familiar.

So, we have:

(i) 15 – (–12) =15 + (additive inverse of –12) =15 + 12 = 27

(ii) (–12) – 21 = (–12) + (additive inverse of 21) = (–12) + (–21) = –33

(iii) (–12) – (–7) = (–12) + (additive inverse of –7) = (–12) + 7 = –5 (as done earlier)

Let the students do the following questions:

1. Find

(i) 36 – 85 (ii) (–25) – 56 (iii) (–37) – (–12)
(iv) (–57) – (–92)

2. Fill in the blanks with >, < , = sign

(i) (–7) + 8 _____ 2 – (–9)

(ii) 5 – 9 ______ (–6) + 2

(iii) 3 + (–9) ______ 7 – 15

3. Fill in the blanks

(i) 15 + _____ = 0

(ii) 8 – ____ = – 2

(iii) 17 – ____ = 19

6.3 MULTIPLICATION

The multiplication of integers involves multiplication of:

(i) two positive integers;

(ii) a positive and a negative integer; and

(iii) two negative integers.

(i) The multiplication of positive integers, i.e. natural numbers is already known to the students. We make use of this for (ii) and (iii).

(ii) Multiplication of a positive and a negative integer can be introduced through grouping and through patterns.

(a) Let us use grouping of integers to find 3 × (–2).

Just as in the case of natural numbers, 3× (–2) will mean adding –2, three times.

3 × (–2) = (–2) + (–2) + (–2) = –6 [Same thing can be shown on the number line also]

(b) Through patterns, 3 × (–2) can be found as

3 × 4 =12

3 × 3 = 9 = 12 – 3

3 × 2 = 6 = 9 – 3

3 × 1 = 3 = 6 – 3

3 × 0 = 0 = 3 – 3

So, 3 × (–1) = 0 – 3 = –3

3 × (–2) = –3 – 3 = –6

Use both the ways and let the student herself get an idea of the value of 3 × (–2).

Give enough practice of such products.

We thus find that

While multiplying a positive and a negative integer, we multiply them as whole numbers and put a minus sign before the product obtained.

Example: (–8) × 5 = – (8 × 5) = –40; 7 × (–9) = – (7 × 9) = –63

In general $a \times (+b) + (-a) \times b = -(a \times b)$ where a and b are positive integers.

(iii) The multiplication of two negative integers can be found by using patterns and the product of a positive and a negative integer.

Let us find (–2) × (–3)

We have: (–2) × 4 = –8

–2 × 3 = –6 = –8 + 2

–2 × 2 = –4 = –6 + 2

–2 × 1 = –2 = –4 + 2

–2 × 0 = 0 = –2 + 2

So, –2 × (–1) = 0 + 2 = 2

–2 × (–2) = 2 + 2 = 4

–2 × (–3) = 4 + 2 = 6

While multiplying two negative integers, first multiply them as whole numbers and then assign a positive sign (+) to the product.

In general, $(-a) \times (-b) = a \times b$, where a and b are positive integers.

The product of two negative integers can be extended further for any number of negative integers, using the properties of associativity and commutativity.

A Special Case

(–1) × (–1) = 1

(–1) × (–1) × (–1) = 1× (–1) = –1

(–1) × (–1) × (–1) × (–1) = (–1) × (–1) = 1

(–1) × (–1) × (–1) × (–1) × (–1) = 1 × (–1) = –1

We find that when (–1) is multiplied odd number of times, the product is –1 and when it is multiplied even number of times, the product is 1.

Note that $3 < 5$ and $3 \times 2 = 6 < 10 = 5 \times 2$ but $3 \times (-2) = -6 > -10 = 5 \times (-2)$

Also $-4 < 7$ and $-4 \times 3 = -12 < 21 = 7 \times 3$ but $(-4) \times (-3) = 12 > -21 = 7 \times (-3)$ and so on. Therefore, an inequality between integers does not change if both sides are multiplied by the same positive integer but it is reversed when both sides are multiplied by the same negative integer.

6.4 DIVISION

Students are already aware that $3 \times (-7) = -21$. So, as in the case of whole numbers, $-21 \div 3 = -7$ and $-21 \div (-7) = 3$. Similarly, as $(-4) \times (-8) = 32$. So, $32 \div (-4) = -8$ and $32 \div (-8) = -4$.

Take more of such examples with different combinations of positive and negative integers.

Finally observing these examples, come to the conclusion that:

(a) When a positive integer is divided by a negative integer or a negative integer is divided by a positive integer, first divide them as whole numbers and then put a minus sign before the quotient.

For example, to find $-18 \div 6$, find $18 \div 6 = 3$. So $-18 \div 6 = -3$.

Similarly, $18 \div (-6) = -3$. So, in either case, we get a negative integer.

In general for any two positive integers a and b, $b \neq 0$, $a \div (-b) = (-a) \div b$.

(b) To divide a negative integer by a negative integer, first divide them as whole numbers and then put a positive sign before the quotient.

In general, for any two positive integers a and b; $b \neq 0$, $(-a) \div (-b) = (a \div b)$

For example, to find $-27 \div (-3)$, first we find $27 \div 3 = 9$. So, $-27 \div (-3) = 9$.

Give sufficient number of examples to the child so that she will observe this process and get a clear idea about it.

As in the case of whole numbers division of integers by zero is not defined.

At this stage, students can be offered the Game 2 'Operations on Integers' of Mathematics Kit for the practice of these concepts while playing.

7. PROPERTIES OF THE OPERATIONS ON INTEGERS

(A) CLOSURE

(i) Addition

Give students different pairs of integers to add. For example $-5 + 3 = -2$ and $8 + (-3) = 5$. Everytime they add the integers, they will again get an integer. After doing additions mechanically their attention should be drawn to the fact that sum of two integers is again an integer. We say that integers are closed under addition, i.e. for any two integers a and b, $a + b$ is always an integer.

(ii) Subtraction

Take different pairs of integers and subtract one from the other. Is the resulting number an integer?

$2 - 3 = -1$ (integer), $-5 - 7 = -12$ (integer)

Finally after giving sufficient number of such examples, it may be brought to the notice of the students that subtraction of an integer from another integer is an integer, i.e. integers are closed under subtraction. In general, $a - b$ is always an integer, where a and b are any two integers.

(iii) Multiplication

Let the students multiply different pairs of integers, as $-2 \times 5 = -10$, $-7 \times (-8) = 56$ and so on.

Ultimately, they will find that for any two integers a and b, $a \times b$ is again an integer, i.e. integers are closed under multiplication.

Before arriving at any of the conclusions ensure that students are given sufficient number of examples so that they develop a notion of that property.

(iv) Division

Let the student divide different pair of integers and observe the quotient.

–15 ÷ 3 = –5 (an integer).

But –7 ÷ 3 is not an integer.

So integers are not closed under division.

(B) COMMUTATIVITY

(i) Addition

5 + (–3) = 2 and (–3) + 5 = 2. Thus, 5 + (–3) = (–3) + 5.

Give several such examples with different combinations of integers and arrive at the result that for any two integers, a and b, we get $a + b = b + a$, i.e. addition is commutative for integers.

(ii) Subtraction

17 – 3 =14, whereas 3 – 17 = –14. So, 17 – 3 ≠ 3 – 17 and so on, i.e. subtraction is not commutative for integers.

(iii) Multiplication

Take different examples, such as –8 × 5 = –40 and 5 × (–8) = –40. Thus, –8 × 5 = 5 × (–8). Take some more such examples to arrive at the result that $a \times b = b \times a$, where a and b are any two integers, i.e. multiplication is commutative for integers.

(iv) Division

Division is not commutative for integers. This can be shown by taking different pairs of integers.

(C) ASSOCIATIVITY

(i) Addition

Take any three integers say –5, 3 and –8. (–5+3) + 8 = –2 + 8 = 6 and –5 + (3 +8) = –5 +11 =6

We find that (–5+3) + 8 = (–5) + (3+8) and so on. Therefore for any integers a, b and c, $(a + b) + c = a + (b + c)$ addition is associative for integers.

(ii) Subtraction

Let us check associative property of subtraction for integers. For integers 2, 3 and 7

(2 – 3) – 7 = –1 – 7 = –8, 2 – (3 – 7) = 2 – (–4) = 2 + 4 = 6, therefore, (2 – 3) – 7 ≠ 2 – (3 – 7).

So, subtraction is not associative for integers.

(iii) Multiplication

As in the case of addition, associative property under multiplication for integers can be verified. That is, for any three integers, a , b and c , $a \times (b \times c) = (a \times b) \times c$.

(iv) Division

Associative property of division for integers does not hold. This can be verified as follows:

(–8 ÷ 4) ÷ 2 = –1, – 8 ÷ (4÷2) = –4. So, (–8÷4) ÷ 2 ≠ –8 ÷ (4÷2)=–4

Property/ Operation	Addition	Subtraction	Multiplication	Division
Closure	✓	✓	✓	×
Commutative	✓	×	✓	×
Associative	✓	×	✓	×

The above properties can be summed up in tabular form

(D) DISTRIBUTIVITY

Take three integers say –7, –8 and 9. Consider –7 × (–8 + 9) = –7 × (1) = –7, (–7) × (–8) + (–7) × 9 = 56 – 63 = –7. So, –7 × (–8 + 9) = (–7) × (–8) + (–7) × 9

Check this for any three integers and verify that for any three integers *a, b* and *c*, $a \times (b + c) = a \times b + a \times c$.

This is called the distributive property of multiplication over addition in integers.

(E) MULTIPLICATIVE IDENTITY

Multiplying any integer by one gives that integer again. Thus, $-15 \times 1 = 1 \times (-15) = -15$, $70 \times 1 = 1 \times 70 = 70$ and so on. We say that 1 is the multiplicative identity for integers.

The utility of using these properties should be brought to the notice of the student by giving many examples. A few are quoted below:

(i) $-8 + 17 - 12 + 13 = [(-8) + (-12)] + (17 + 13)$ (Associativity and commutativity for addition)
$= -20 + 30 = 10$

(ii) $-5 \times 76 \times 20 = (-5 \times 20) \times 76$ (Associative and commutative property for multiplication)
$= -100 \times 76 = -7600$

Allow the students to think on the problems. If they come out with a way which is mathematically correct then they should be encouraged to do so.

(iii) $-18 \times 27 = -18 \times (20 + 7)$
$= -18 \times 20 + (-18) \times 7$ (Distributive)
$= -360 + (-126) = -486$

Common Errors

(i) $(-1) + (-1) = 0$
$(-1) \times (-1) = -1^2 = -1$
$(1) - (-1) = 0$, $(-1) \div (-1) = -1$

(ii) $-1 > 0$

(iii) $-3 > -2$

(iv) 4 more than (-1) is $4 - (-1) = 5$

(v) 5 is greater than 3, therefore $-5 > -3$

(vi) Additive inverse of -5 is -5 or additive inverse of -5 is $\frac{1}{5}$

(vii) Multiplicative inverse of 0 is $\frac{1}{0}$

(viii) $-10 + 3 = 10 - 3$

(ix) $(-3) - (-3) = -6$

(x) $-3 - 5 = -2$

(xi) $3 - 5 = 2$

(xii) $(-1) \times (-1) \times (-1) \times (-1) \times (-1) = 1$

(xiii) The larger integer among –8 and –1 is –8.

(xiv) The smaller integer among –8 and –1 is –1.

You may evaluate students through the following exercise.

Exercise

1. Find the sum of the following:
   (i) $(-370) + 200$ (ii) $301 + (-67)$
   (iii) $(-87) + (-16)$ (iv) $37 + (-15) + (-23) + 18$

2. Find
   (i) $-9 - 18 - (-21)$
   (ii) $62 - (-39) - (-5)$

3. Use the sign of $>$, $<$ or $=$ in the blank space
   (i) $(-11) + (-7)$ ...... $(-11) - (-7)$
   (ii) $(-24) + 16 - 15$ .... $(-26) - 12 - (-85)$

4. Write down a pair of integers whose
   (a) sum is –4 (b) difference is 5
   (c) sum is 0

5. Fill in the blanks
   (i) $[26 + (-5)] +$...........$= 26 + [(-5) + (-3)]$
   (ii) $-87 +$ .... $= -87$
   (iii) $67 + ($...............$) = 0$

6. Find
   (i) $276 \times (-3) \times 0$
   (ii) $(-15) \times 8 \times (-9)$

(iii) (–7) × (–3) × (–2) × (–1)

7. Write the integer whose product with (–1) is
   (i) –35
   (ii) 47
   (iii) 0
8. Find the product using suitable properties and also name the properties being used
   (i) 35 × (–7) + (–7) × 65
   (ii) 4 × (–76) × 25
   (iii) (–23) × 99 – 23
9. Evaluate
   (i) [(–48) ÷12] ÷ 4
   (ii) 0 ÷ (–7)
   (iii) (–29) ÷ [(–28) + (–1)]
10. Write five pairs of integers (*a*, *b*) such that *a* ÷ *b* = –2 [one such pair is (12 – 6)]
11. At Shimla, temperature was –1°C on Friday and was increased by 3°C on Saturday. What was the temperature of Shimla on Saturday? On Sunday, it decreased by 6°C. What was the temperature on this day?
12. In a class test containing 12 questions, 3 marks are given for every correct answer and (–2) marks are given for every incorrect answer and 0 for questions not attempted. Rahul attempts all the questions but only 8 of his answers are correct. What is his score in the test?

Unit 4

Fractions and Decimals

Structure

- Introduction
- Main Concepts and Sub-concepts
- Objectives
- Teaching Points

  1. Fraction as a part of whole
  2. Fractions on a number line
  3. A Fraction

     3.1. Fraction as division

  4. Proper fractions
  5. Improper and mixed fractions
  6. Equivalent fractions

     6.1 Understanding of equivalent fractions

     6.2 Simplest or lowest form of a fraction

  7. Like and Unlike Fractions
  8. Comparing Fractions

     8.1 Comparing like fractions

     8.2 Comparing unlike fractions

Introduction

Fractions are, in fact, a child's first excursion into abstract mathematics. The teaching of fractions is spread roughly over Classes II to VII. In the Classes II to V, more or less, students' learning is mainly focussed on acquiring the vocabulary of fractions and using it for descriptive purposes. It is only in Class VI and upwards that a systematic learning of the mathematics of fractions takes place. In these classes, students begin to put the isolated bits of information they have acquired into a mathematical framework and learn how to compute extensively with fractions. So, when students reach Class VI, they have to learn a precise mathematical concept of a fraction and make a logical sense of the fraction skill. And that's why, as a teacher, we have to nurture the fraction concept at Class VI level very carefully. Students' fear of fractions is well documented and at the same time there is no such pervasive fear in the early vocabulary-acquiring stage. In the second stage, however, this fear becomes real and seems to develop around the time they learn how to add fractions using the least common denominator. Thus a caution for proper development in the above mentioned sequence is needed. In learning to deal with the Mathematics of natural numbers in Classes I to IV, children always have a natural reference point: their fingers. The modeling of whole numbers on one's finger is both powerful and accurate. After the initial introduction of fractions with the use of part of a whole (refer to Block Annexure-unit 4 and 5), if we as a teacher could develop a proper understanding of fractions and operations on fractions with the reference of number line, it would become a powerful tool for the students in much wider understanding.

Main Concepts and Sub-concepts

- Fraction—A Conceptual Understanding
- Fraction on the Number Line
- Kinds of Fractions
    - Proper fractions
    - Improper and mixed fractions
- Equivalent Fractions
- Comparison of Fractions
- Operations on Fractions
    - Addition and subtraction of fractions
    - Multiplication of fractions
    - Division of fractions
- Decimals and Operations on Decimals.

Objectives

After teaching this unit, the student can

- understand the meaning of a fraction or fractional number;
- identify numerator and denominator in a fraction;
- represent a fraction pictorially and on the number line;

- make equivalent fractions and identify equivalent fractions from a given collection;
- identify largest and smallest fractions in a given collection;
- arrange fractions in descending and ascending orders;
- explain fraction as an operation 'of' or 'division';
- do operations on fractions (addition, subtraction, multiplication and division);
- apply the knowledge of fractions in solving problems; and

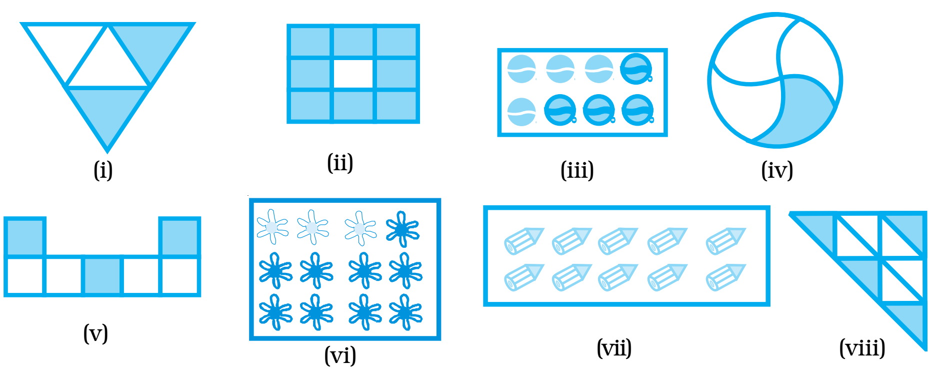
  understand the meaning of decimal and perform 'addition' and 'subtraction' on decimals.

Teaching Points

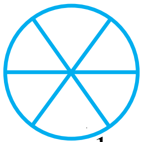
1. FRACTION AS A PART OF A WHOLE

Fraction should be introduced to the children through the concrete models based on area concepts. This provides the visual empowerment among the children. As per the student's prior knowledge of vocabulary of fractions, at least a couple of periods should be utilized to engage them with pictorial activities and exercises as given in the Class VI, Mathematics, NCERT.

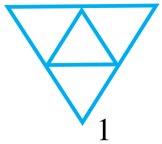
Teacher may show the following diagrams to students and discuss among children about the fractional part (shaded) in each of them.



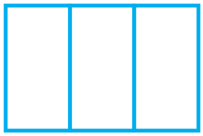
Let the students shade the part represented by the given fraction in each of the following:



(i) $\frac{1}{6}$

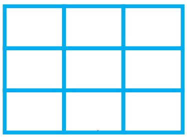


(ii) $\frac{1}{4}$



(iii) $\frac{1}{3}$

(iv) $\frac{3}{4}$



(v) $\frac{4}{9}$

At this stage children can be provided a concrete experience of this concept by the Activity 2 'Exploring Fractions' of Mathematics kit.

2. FRACTION ON A NUMBER LINE

After ensuring that students have become familiar with the vocabulary and have a visual understanding of fractions, they have to move towards abstraction very carefully with the use of number line as follows:

We begin with the number line. So on a line which is (usually chosen to be) horizontal, we pick a point and designate it as 0. We then choose another point to the right of 0 and by reproducing the distance between 0 and this point, we mark some more points at equal distances. Next we denote all these points by the non-zero whole numbers 1, 2, 3 ... in the usual manner. Thus all the whole numbers 0, 1, 2, 3... are now displayed on the number line as equi-spaced points increasing to the right of 0, as shown in Fig. 4.1.

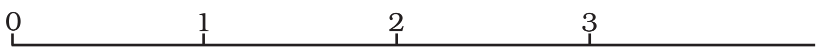


Fig. 4.1

Fractions are a special class of numbers constructed in the manner below:

If a and b are two points on the number line, with a to the left of b, we denote the segment from a to b by $[a, b]$.

The points a and b are called the **endpoints** of $[a, b]$. The special case of the segment [0, 1] occupies a distinguished position in the study of fractions; it is called the **unit segment.** The point 1 is called the **unit.** As mentioned above, 0 and 1 determine the points we call the whole numbers. So if 1 stands for an orange, 5 would be 5 oranges, and if 1 stands for 5 kilograms of rice, then 6 would be 30 kilograms of rice. And so on.

We take as our 'whole' the unit segment [0,1]. The fraction $\frac{1}{3}$ is, therefore, one-third of the whole, i.e. if we divide [0,1] into 3 equal parts, $\frac{1}{3}$ stands for one of the parts. One obvious example is the thickened segment below, and we use the right end-point of this segment as the standard representation of $\frac{1}{3}$ (Fig.4.2).

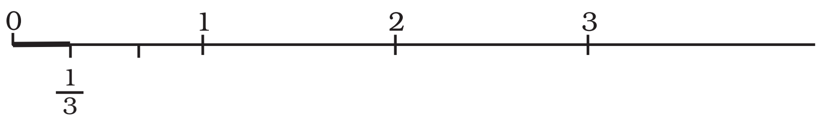


Fig. 4.2

We next divide, not just [0, 1], but every segment with their end points as two consecutive whole numbers—[0, 1], [1, 2], [2, 3], [3, 4], etc.—into three equal parts. Now let us check how we will represent the fraction $\frac{10}{3}$. Intuitively, it stands for '10 copies of one–thirds', i.e. $10\times\frac{1}{3}$ and therefore has the **standard representation** consisting of 10 adjoining short segments (segments of $\frac{1}{3}$) starting with 0. Thus, we may identify this standard representation of $\frac{10}{3}$ with its right endpoint as shown in Fig. 4.3

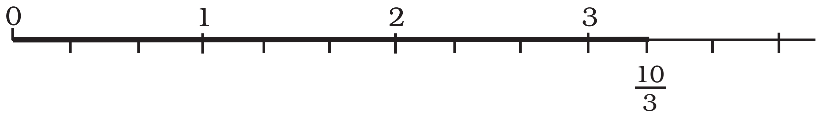


Fig. 4.3

Basically, these are exactly the fractions with denominator equal to 3.

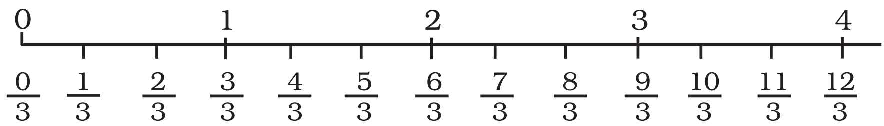


Fig. 4.4

Similarly, the fractions with denominator 5 can be represented on the number line as shown in the following figure (Fig.4.5):

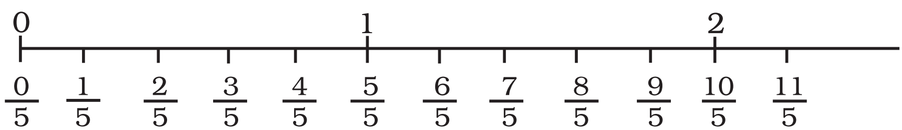


Fig. 4.5

Here we are dividing each of the segments [0,1], [1, 2], [2, 3], ..., into 5 equal parts. After the initial introduction of fraction as part of a whole, students should be trained in the mathematical way of fraction.

3. A FRACTION

A fraction means a part of a group or of a region or more accurately, a fraction say ($\frac{1}{3}$) is the first point to the right of 0 when one unit is divided in three equal parts. Here, again the name 'denominator' and 'numerator' should be made clear in student's minds, while making the divisions on the number line, i.e. the denominator shows the number of parts into which the **unit segment** is divided and the numerator shows how many parts have been taken starting from 0.

Thus, numbers of the form $\frac{a}{b}$, where a and b are whole numbers and $b \neq 0$, are called fractions.

FRACTION AS DIVISION

For any two whole numbers m and n, $n \neq 0$, we define **the division of *m* by *n*** as follows:

$m \div n$ is the length of one part when a segment of length m is partitioned into n equal parts.

Why this definition? Because students coming in Class VI only know about the meaning of '9 divided by 3', '28 divided by 7', or in general, 'm divided by n when m is a multiple of n'. But now we are talking about the division of arbitrary positive integers such as '5 divided by 7' or '28 divided by 9'. Such divisions are conceptually distinct from the earlier concept of 'division'. And, thus with this definition of division, we have that .

4. PROPER FRACTIONS

While representing the fractions on the number line, the fractions which lie left of 1 are proper fractions. Thus, in a proper fraction the denominator is always greater than the numerator.

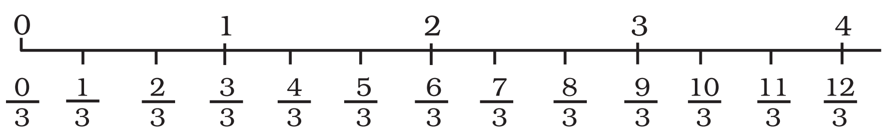


Fig. 4.6

For example, $\frac{1}{3}, \frac{2}{3}$ are proper fractions and lie to the left of 1. (See Fig.4.6)

Similarly, $\frac{5}{7}, \frac{9}{13}$ are also proper fractions and fractions $\frac{12}{7}, \frac{8}{3}$ are not proper fractions.

5. Improper Fractions and Mixed Fractions

The fractions which are coming after 1 or are to the right of 1 on the number line are improper fractions. Thus, in an improper fraction, the denominator is always smaller than the numerator. Here it should be noted that an activity as given on page number 138, Class VI, Mathematics textbook of NCERT or similar activity should be discussed or performed to link the improper fractions and mixed fractions with student's daily life.

Basically, it is important to realize that a mixed fraction is a combination of a whole and a proper fraction. Mixed fraction = whole number + proper fraction, i.e. $a\frac{b}{c} = a + \frac{b}{c}$.

For example, $1\frac{1}{3} = 1 + \frac{1}{3}$. Improper fraction and mixed fraction can be made from each other. A visual justification of this statement is needed at Class VI stage.

6. Equivalent Fractions

This is an important concept related to fractions. For a fraction $\frac{m}{n}, \frac{km}{kn}$ is an equivalent fraction if $k \neq 0$, i.e. $\frac{m}{n} = \frac{km}{kn}$.

For example, for a fraction $\frac{4}{3}$, $\frac{5\times4}{5\times3}$ is an equivalent fraction, i.e. $\frac{4}{3}$ and $\frac{20}{15}$ are equivalent fractions. Let us see this fact on the number line. First locate $\frac{4}{3}$ on the number line (Fig.4.7):

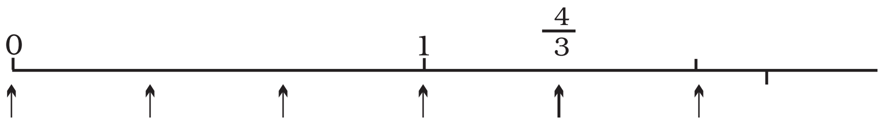


Fig. 4.7

We divide each of the segments between consecutive points in the sequence of one-thirds into 5 equal parts. Then each of the segments [0, 1], [1, 2],.... is now divided into 15 equal parts and, in an obvious way, we have obtained the sequence of one-fifteenth on the number line

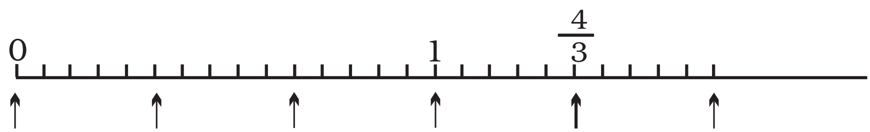


Fig. 4.8

The point $\frac{4}{3}$, being the 4th point in the sequence of one-thirds, is now the 20th point in the sequence of one-fifteenth (20 being equal to 5 × 4) (Fig. 4.8). Thus $\frac{4}{3} = \frac{5 \times 4}{5 \times 3}$.

6.1 UNDERSTANDING OF EQUIVALENT FRACTIONS

At the stage of Class VI, some more visual justification of equivalent fractions is needed. On Page 142 of Class VI, Mathematics, NCERT , a pictorial representation of the above fact has been shown. Students should be engaged in such activities for understanding the concept of equivalent fractions.

Example: Identify the fractions in each of the figures of Fig. 4.9. Are these fractions equivalent?

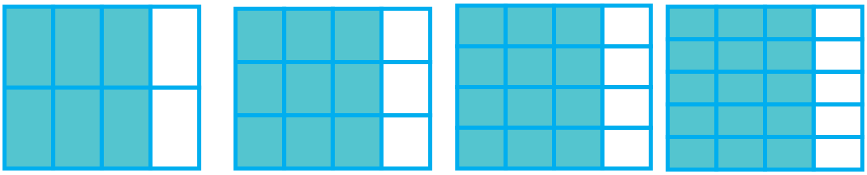


Fig. 4.9

Similarly, by following activity one can check that *To find an equivalent fraction, we may multiply/divide both the numerator and the denominator by the same non-zero number.*

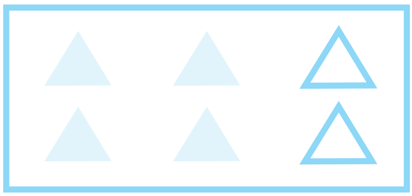


Fig. 4.10

In Fig. 4.10 out of 6 triangles 4 are shaded and it is representing the fraction $\frac{4}{6}$. Again on regrouping, we have 3 groups, out of which 2 are shaded as in Fig. 4.11.

Fig. 4.11

Thus, the same fraction is equal to the fraction $\frac{2}{3}$. Thus $\frac{4}{6}=\frac{2}{3}=\frac{4 \div 2}{6 \div 2}$. Students should be made to note that by multiplying or dividing the numerator and denominator of a fraction by the same non-zero number does not change the value of the fraction.

All such fractions are known as **equivalent fractions.**

Again here Activity 2 'Exploring Fractions' of Mathematics kit can be performed for concrete experiences.

In terms of finding the equivalent fractions, 'cross multiplication' has been introduced. The important point here is that this property should be transacted among the children through patterns. One pattern has been given on page 144 of Class VI Mathematics, NCERT.

Let us see one example:

Example: Find the equivalent fraction of $\frac{2}{9}$ with denominator 63.

Solution: We have $\frac{2}{9} = \frac{\square}{63}$

For this, we should have, $9 \times \square = 2 \times 63$.

But $63 = 7 \times 9$, So $9 \times \square = 2 \times 7 \times 9 = 14 \times 9$ $= 9 \times 14$

or $9 \times \square = 9 \times 14$

By comparison, $\square = 14$. Therefore, $\frac{2}{9} = \frac{14}{63}$.

6.2 SIMPLEST OR LOWEST FORM OF A FRACTION

We have seen that equivalent fraction can be found by dividing the numerator and the denominator of a fraction by the same non-zero number. Basically, this number should be a common factor of the numerator and the denominator both.

A fraction is said to be in the simplest (or lowest) form if its numerator and denominator have no common factor except 1.

Thus, the shortest way to find the equivalent fraction in the simplest form is to find the HCF of the numerator and denominator of the given fraction and then divide both of them by their HCF.

Students may note that a fraction is said to be in the simplest form, if the HCF of its numerator and denominator is 1.

To motivate children, a game has been suggested on page 145 of Class VI Mathematics, NCERT. Similar situations can be provided to generate more interest among the children.

A Game

The equivalent fractions given here are quite interesting. Each one of them uses all the digits from 1 to 9 once!

$$\frac{2}{6}=\frac{3}{9}=\frac{58}{174}$$

$$\frac{2}{4}=\frac{3}{6}=\frac{79}{158}$$

Try to find two more such equivalent fractions.

7. LIKE AND UNLIKE FRACTIONS

Fractions with same denominators are called **like fractions**. For example, $\frac{1}{15}, \frac{2}{15}, \frac{5}{15}, \frac{12}{15}$ are all like fractions. And, therefore, the fractions which have different denominators are called **unlike fractions**. For example, $\frac{1}{15}$ and $\frac{2}{5}$ are unlike fractions. Through the concept of like and unlike fractions, we can introduce the concept of comparison of fractions easily among the students.

8. Comparing Fractions

Consider $\frac{1}{2}$ and $\frac{1}{3}$ as shown in the following Fig. 4.12. The portion of the whole corresponding to $\frac{1}{2}$ is clearly larger than the portion of the same whole corresponding to $\frac{1}{3}$.

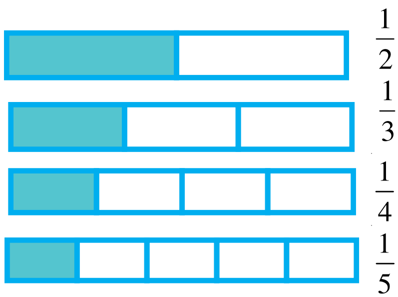


Fig. 4.12

Initially the concept of comparison of fractions should be realized through such pictures. Then for the systematic development of the comparison of fractions, it is better if comparison of like fractions is given first.

8.1 COMPARING LIKE FRACTIONS

In like fractions, for example, $\frac{1}{15}, \frac{2}{15}, \frac{5}{15}, \frac{12}{15}$ denominators of all these fractions are same. Visually students can be made to realize that in like fractions, the fraction with the greatest numerator will be the largest among all the given like fractions. It should also be mathematically justified as follows. Divide the unit segment [0, 1] in 15 equal parts, then look at the position of all these like fractions on the number line. We see that the fraction $\frac{12}{15}$ represents the longest distance from 0 in the right side. Therefore, $\frac{12}{15}$ is the greatest and $\frac{1}{15}$ is the smallest in the given like fractions (Fig. 4.13).

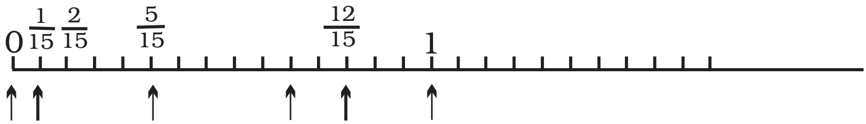


Fig. 4.13

Thus, if we have two fractions with the same denominator, the fraction with the greater numerator is greater.

8.2. COMPARING UNLIKE FRACTIONS

Let us consider a pair of unlike fractions, say, $\frac{1}{3}$ and $\frac{1}{5}$, in which the numerator is the same. Which is greater: $\frac{1}{3}$ or $\frac{1}{5}$?

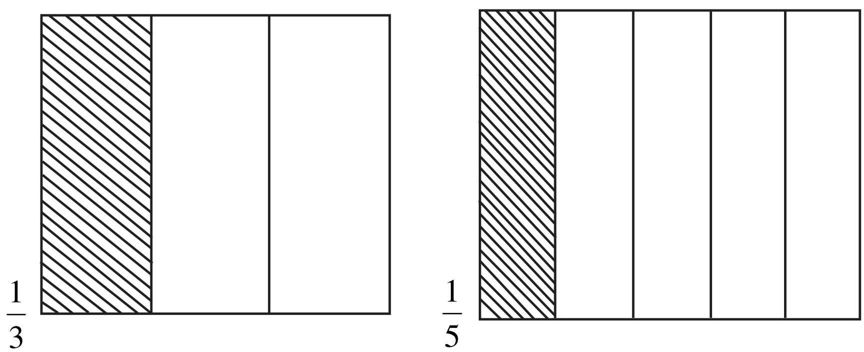


Fig. 4.14

In Fig. 4.14, the shaded part representing $\frac{1}{3}$ is larger than the shaded part representing $\frac{1}{5}$. Thus, $\frac{1}{3} > \frac{1}{5}$. In the same way, we can say $\frac{2}{3} > \frac{2}{5}$ We can see from these examples **that if the numerator is the same in two fractions, the fraction with the smaller denominator is greater of the two.**

Example: Let us arrange $\frac{2}{1}, \frac{2}{13}, \frac{2}{9}, \frac{2}{5}, \frac{2}{7}$ in increasing order.

Solution: All these fractions are unlike, but their numerator is same. Hence, in each case, larger the denominator, smaller is the fraction. Therefore, the arrangement in increasing order is

$$\frac{2}{13}, \frac{2}{9}, \frac{2}{7}, \frac{2}{5}, \frac{2}{1} \text{ i.e., } \frac{2}{13} < \frac{2}{9} < \frac{2}{7} < \frac{2}{5} < \frac{2}{1}.$$

Now we will proceed for the general case. Suppose we want to compare $\frac{2}{3}$ and $\frac{3}{4}$. Here numerators and denominators of both fractions are different. Since we already know the comparison among like fractions, i.e. fractions with the same denominators, we convert both the given fractions into like fractions. In this regard, we find the common multiple of the denominators of both the fractions. Since common multiple of 3 and 4 is 12, we get like fractions of given fractions as $\frac{8}{12}$ and $\frac{9}{12}$, respectively. Thus $\frac{3}{4} > \frac{2}{3}$, because $\frac{9}{12} > \frac{8}{12}$.

Since it is more convenient to work with smaller numbers, so the common multiple that we take to make the fractions like fractions should be as small as possible. This is why the LCM of the denominators of the given fractions is preferred as the common denominator.

Thus, given two fractions $\frac{k}{l}$ and $\frac{m}{n}$, we say that $\frac{m}{n} < \frac{k}{l}$, i.e. $\frac{m}{n}$ is less than $\frac{k}{l}$ or $\frac{k}{l}$ is bigger than $\frac{m}{n}$, if the point $\frac{m}{n}$ is to the left of the point $\frac{k}{l}$ on the number line (Fig. 4.15).

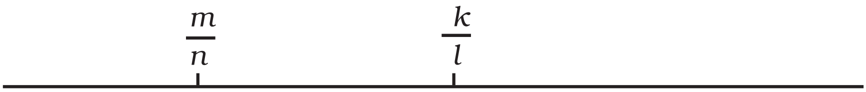


Fig. 4.15

Basically, the **fundamental fact** is that

Any two fractions may be expressed as fractions with the same denominator.

The reason is simple: If the fractions are $\frac{m}{n}$ and $\frac{k}{l}$, then because of equivalent fractions, we have $\frac{m}{n} = \frac{ml}{nl}$ and $\frac{k}{l} = \frac{nk}{nl}$. Since the denominators of both the fractions are same, it is now easy to compare these fractions.

9. Operations on Fractions

9.1 ADDITION OF FRACTIONS

The addition of fractions cannot be different, conceptually, from the addition of whole numbers because every whole number is also expressed as a fraction. How do we add whole numbers when whole numbers are considered points on the number line? Consider, for example, the addition of two whole numbers 4 and 7. In terms of the number line, this is just the total length of the two segments joined together end-to-end, one of length 4 and the other of length 7, which is, of course, 11 as shown in Fig. 4.16.

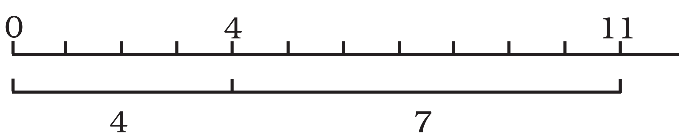


Fig. 4.16

We call this process the 'sum up' of the two segments.

Similar to this process, for two fractions $\frac{k}{l}$ and $\frac{m}{n}$, we define their sum $\frac{k}{l}+\frac{m}{n}$ by $\frac{k}{l}+\frac{m}{n}$ = the length of two summed up segments, one of length $\frac{k}{l}$ followed by one of length $\frac{m}{n}$ (Fig.4.17).

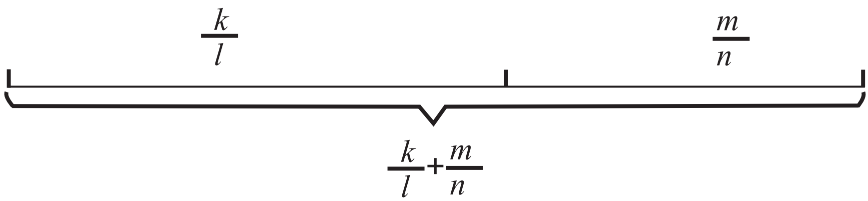


Fig. 4.17

It is an immediate consequence of the definition that $\frac{k}{l}+\frac{m}{l}=\frac{k+m}{l}$ because both sides are equal to the length of $k + m$ copies of $\frac{1}{l}$. More explicitly, the left side is the length of k copies of $\frac{1}{l}$ combined with m copies of $\frac{1}{l}$ and is therefore the length of (k+m) copies of $\frac{1}{l}$ which is exactly the right side. Basically, before the mathematical realisation of above addition fact of like fractions, some more visual demonstration may be provided to the children. One of the demonstrations is given on page 155 of Class VI Mathematics, NCERT which is as follows:

Take a 7 x 4 grid sheet (Fig. 4.18). The sheet has seven boxes in each row and four boxes in each column.

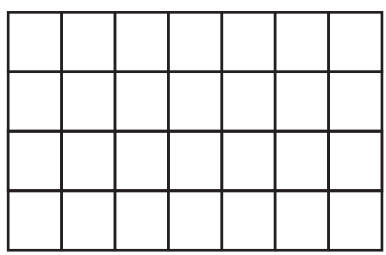


Fig. 4.18

How many boxes are there in total?

Colour five of its boxes in green.

What fraction of the whole is the green region?

Now colour another four of its boxes in yellow.

What fraction of the whole is this yellow region?

What fraction of the whole is coloured altogether?

Does this explain that $\frac{5}{28}+\frac{4}{28}=\frac{9}{28}$.

For fractions like $\frac{k}{l}$ and $\frac{m}{n}$ with unequal denominators, to find their sum, we proceed as follows:

By fundamental fact, we can have equivalent forms of these fractions with the same denominator, i.e.

$$\frac{k}{l}=\frac{nk}{nl} \text{ and } \frac{m}{n}=\frac{ml}{nl}.$$

Thus, $\frac{k}{l}+\frac{m}{n}=\frac{nk}{nl}+\frac{ml}{nl}=\frac{nk+ml}{nl}$

We can summarise addition of fractions as

(i) Sum of like fractions = $\frac{\text{Sum of their numerators}}{\text{Common denominator}}$

(ii) For sum of unlike fractions, change the given fractions into equivalent like fractions and then add.

Now, consider the addition of $2\frac{4}{5}+3\frac{5}{6}$. One can do this routinely by converting the mixed fractions into improper fractions.

$$2\frac{4}{5}+3\frac{5}{6}=\frac{10+4}{5}+\frac{18+5}{6}=\frac{14}{5}+\frac{23}{6}=\frac{14\times6+23\times5}{5\times6}=\frac{84+115}{30}=\frac{199}{30}=6\frac{19}{30}$$

However, there is another way to do the computation which is as follows:

$$2\frac{4}{5}+3\frac{5}{6}=2+\frac{4}{5}+3+\frac{5}{6}=5+\frac{4}{5}+\frac{5}{6}$$

Now, $\frac{4}{5}+\frac{5}{6}=\frac{4\times6}{5\times6}+\frac{5\times5}{6\times5}=\frac{4\times6+5\times5}{5\times6}=\frac{24+25}{30}=\frac{49}{30}=\frac{30+19}{30}=1+\frac{19}{30}$

Thus, $5+\frac{4}{5}+\frac{5}{6}=5+1+\frac{19}{30}=6+\frac{19}{30}=6\frac{19}{30}$

9.2 SUBTRACTION OF FRACTIONS

Let us consider two fractions $\frac{k}{l}$ and $\frac{m}{n}$. Suppose $\frac{k}{l}>\frac{m}{n}$.

Then the segment of length $\frac{k}{l}$ is longer than the segment of length $\frac{m}{n}$. The subtraction $\frac{k}{l}-\frac{m}{n}$ is, by definition, the length of the remaining segment when a segment of length $\frac{m}{n}$ is taken out from one end of a segment of length $\frac{k}{l}$.

Using the same reasoning as in the case of addition, we get

$$\frac{k}{l}-\frac{m}{n}=\frac{kn-ml}{nl}$$

To start with, a visual demonstration has been given as 'Finding the balance' on page 156 in Class VI, Mathematics, NCERT as follows:

We can see the expression $\frac{5}{6}-\frac{2}{6}=\frac{5-2}{6}=\frac{3}{6}$ as follows (Fig. 4.19)

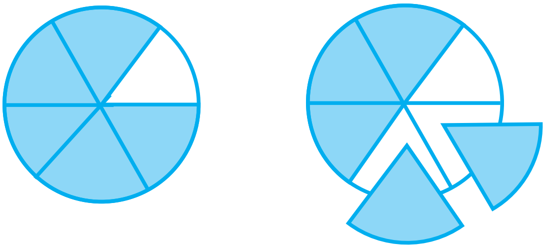


Fig. 4.19

We can summarise subtraction of fractions as

(i) Difference of like fractions $= \dfrac{\text{Difference of their numerators}}{\text{Common denominator}}$

(ii) For the difference of unlike fractions, change the fractions into equivalent like fractions and then subtract.

10. MULTIPLICATION OF FRACTIONS

10.1 MULTIPLICATION OF A FRACTION BY A WHOLE NUMBER

A visual demonstration for this multiplication is given on page number 32 of Class VII Mathematics, NCERT as follows:

Observe the pictures at the left. Each shaded part is $\frac{1}{4}$ part of a circle. How much will the two shaded parts represent together? They will represent $\frac{1}{4}+\frac{1}{4}=2\times\frac{1}{4}$.

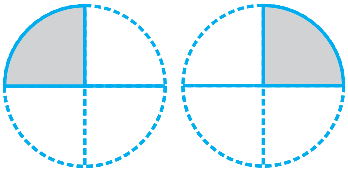


Fig. 4.20

Combining the two shaded parts, we get following Fig. 4.21. What part of a circle does the shaded part in the figure represent? It represents $\frac{2}{4}$ part of a circle.

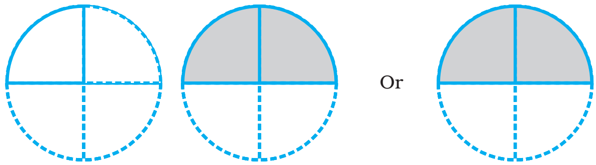


Fig. 4.21

The shaded portions in earlier Fig. 4.20 taken together are the same as the shaded portion in the later Fig. 4.21, i.e. the following Fig. 4.22

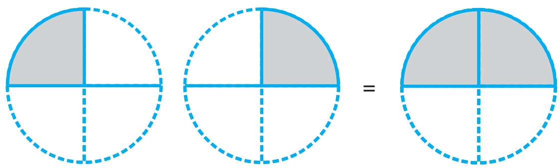


Fig. 4.22

The important point here is to recognize the pattern that while multiplying a whole number with a proper or an improper fraction, we multiply the whole number with the numerator of the fraction, keeping the denominator same, for example: $3\times\frac{1}{4}=\frac{3}{4}$.

10.2 MULTIPLICATION OF A FRACTION BY A FRACTION

In general, we define $\frac{m}{n}$ of a number to mean a total of m parts when that number is partitioned into n equal parts. More explicitly, if the number is a fraction $\frac{k}{l}$, then we partition the segment $[0, \frac{k}{l}]$ into n parts of equal length, and $\frac{m}{n}$ of $\frac{k}{l}$ is the length of the 'sum up' of m number of these parts. Then we define the product or multiplication of two fractions by

$$\frac{k}{l}\times\frac{m}{n}= \frac{k}{l} \text{ of a segment of length } \frac{m}{n}$$

We have the reason behind this computation $\frac{k}{l} \times \frac{m}{n} = \frac{km}{ln}$.

Let us partition $[0, \frac{m}{n}]$ into l equal parts. By the definition of the product $\frac{k}{l} \times \frac{m}{n}$, it suffices to check that the length of k 'summed up' parts is $\frac{km}{ln}$. By equivalent fractions,

$$\frac{m}{n} = \frac{lm}{ln} = \frac{\overbrace{m + \ldots + m}^{l \text{ times}}}{ln} = \underbrace{\frac{m}{ln} + \ldots. + \frac{m}{ln}}_{l \text{ times}}$$

This directly exhibits that $\frac{m}{n}$ is the sum up of l parts, each part of length $\frac{m}{ln}$. The length of k 'summed up' parts is thus $\frac{km}{ln}$, as desired.

A visual demonstration with activity for this multiplication is given on pages 37–40 of Class VII Mathematics, NCERT. Through patterns, it has been realised that multiplication of two fractions is $\frac{\text{Product of Numerators}}{\text{Product of Denominators}}$, for example, $\frac{1}{2} \times \frac{5}{3} = \frac{1 \times 5}{2 \times 3} = \frac{5}{6}$. Thus, in the beginning these activities should be performed and while teaching the multiplication of fractions such connections between visuals and computations should be made.

11. DIVISION OF FRACTIONS

We teach children that $\frac{36}{9} = 4$ because 4 is the whole number such that $4 \times 9 = 36$. This then is the fact that 36 divided by 9 is that whole number which when multiplied by 9, gives 36. In symbols, we may express the foregoing as follows: $\frac{36}{9}$ is by definition the number k which satisfies $k \times 9 = 36$.

Similarly, $\frac{7}{4}$ is the number which satisfies $\frac{7}{4} \times 4 = 7$.

In general for given whole numbers a and b, with $b \neq 0$ and a being multiple of b, the division of a by b, (in symbols), is the whole number c such that the equality $cb = a$ holds.

The preceding definition of division among whole numbers is important for the understanding of division among fractions, because once we replace 'whole number' by 'fraction', this will be essentially the definition of the division of fractions. However, there is a caution. The definition in case for whole numbers a and b, the division $\frac{a}{b}$ makes sense only when a is a multiple of b. The first task in approaching the division of fractions is to show that, if a and b are fractions, $\frac{a}{b}$ would always makes sense so long as b is non-zero. It can be seen like this, let $a = \frac{k}{l}$ and $b = \frac{m}{n}$, then the fraction c defined by $c = \frac{kn}{lm}$ clearly satisfies $a = cb$. (Since b is non-zero, m is non-zero. Therefore $lm \neq 0$ and this fraction c makes sense).

Two fractions are said to be the **reciprocal** of each other, if their product is 1.

For Example

(i) and $\frac{5}{2}$ are reciprocals of each other as $\left(\frac{2}{5} \times \frac{5}{2}\right) = 1$.

(ii) 9 and $\frac{1}{9}$ are reciprocals of each other as $\left(\frac{9}{1} \times \frac{1}{9}\right) = 1$.

In general, if $\frac{a}{b}$ is a non-zero fraction, then its reciprocal is $\frac{b}{a}$. It may be emphasised that reciprocal of 0 does not exist.

The students may be encouraged to do the 'Think, Discuss and Write' on page 44 of Class VII Mathematics, NCERT. We teach the children that when we have to divide a fraction by another whole number or fraction, the first fraction is multiplied by the reciprocal of the second one.

Thus, in general $\frac{a}{b} \div c = \frac{a}{b} \times \frac{1}{c}$

and $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$

Example: Divide (i) $\frac{2}{3}$ by $4 = \frac{2}{3} \div 4 = \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$

(ii) $\frac{4}{7}$ by $\frac{8}{3} = \frac{4}{7} \div \frac{8}{3} = \frac{4}{7} \times \frac{3}{8} = \frac{3}{14}$

Students may now be encouraged to do both 'Try These' given on page 45 of Class VII, Mathematics, NCERT.

Teach the students that to perform operations on mixed fractions, first convert them into improper fractions and then solve.

Students will now be able to solve all the questions of Exercise 2.4, Page 46, Class VII, Mathematics, NCERT. For more exploration for division of fractions, you may refer to Block Annexure, Unit 5 'Division of Fractions'.

12. DECIMALS

Those fractions whose denominators are 10, 100, 1000 and so on, e.g. $\frac{14}{10}, \frac{24}{100}, \frac{580}{100}, \frac{3}{1000}$, are called decimal fractions or decimal but they are better known in a different notation. These fractions are therefore abbreviated as 1.4, 0.24, 5.80, 0.003, respectively.

A decimal number has two parts:

(i) Whole number part (part to the left of the decimal)

(ii) Decimal part (part to the right of the decimal)

The number of digits in the decimal part gives the number of decimal places.

The rationale of the notation is clear: the number of digits to the right of the so-called decimal point keeps track of the number of zeros in the respective denominators, 1 in 1.4, 2 in 0.24, 2 in 5.80 and 3 in 0.003. In this notation, these numbers are called finite or terminating decimals. In the present context, we will omit any mention of 'finite' or 'terminating' and just say 'decimals'. Notice the convention that, in order to keep track of the 3 zeros in $\frac{3}{1000}$, two zeros are added to the left of 3 to make sure that there are 3 digits to the right of the decimal point in 0.003. Note also that the 0 in front of the decimal point is only for the purpose of clarity, and is optional. One should know that 5.8 is same as 5.80. This can be seen as the first application of equivalent fractions. We have $\frac{580}{100}=5.80$. We can see that 5.80 = 5.8 as follows and, more generally, one can add or delete zeros to the right end of the decimal point without changing the decimal. Indeed, $5.80=\frac{580}{100}=\frac{58\times10}{10\times10}=\frac{58}{10}=5.8$, here the last equality makes use of equivalent fractions. The reasoning is of course valid in general, e.g.

$$12.7=\frac{127}{10}=\frac{127\times100}{10\times100}=\frac{12700}{1000}=12.700$$

A visual demonstration of one-tenth, one-hundredth is given on page 169 of Class VI Mathematics, NCERT as follows:

Take a square and divide it into ten equal parts. What part is the shaded rectangle of this square?

It is $\frac{1}{10}$ or one-tenth or 0.1 (see Fig. 4.23).

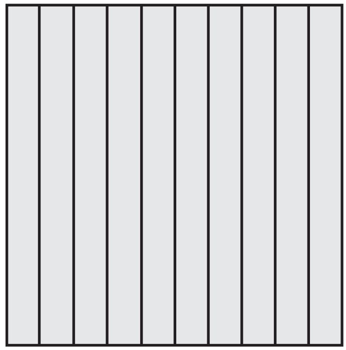


Fig 4.23

Now divide each such rectangle into ten equal parts.

We get 100 small squares as shown in Fig. 4.24.

Then what fraction is each small square of the whole square?

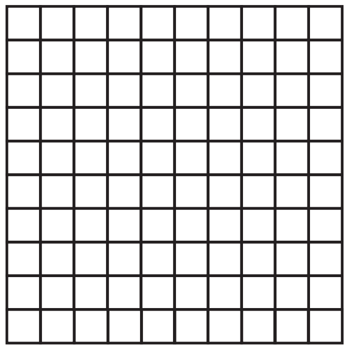


Fig 4.24

Each small square is $\left(\frac{1}{100}\right)$ or one-hundredth of the whole square. In decimal notation, we write $\left(\frac{1}{100}\right)$ = 0.01 and read it as zero point zero one.

While introducing the decimal concept these visual activities should be performed and connections between visuals and numerical representations of decimal should be established.

Similarly comparison of decimals has been given in terms of place value system on page number 174, Class VI, Mathematics, NCERT Textbook and at the same time this can also be dealt with using fraction concept. Then follows the operations on decimals. Using place value system the addition and subtraction of decimals can be performed as done for whole numbers. However for the multiplication and division of decimals, it has been mentioned that concept should be developing using visual representations before involvement in calculations through the fraction concept. For the visual demonstration please refer to pages 48–49 of Class VII Mathematics, NCERT.

Activity-3 'Abacus' of Mathematics Kit has to be utilised for this concept.

Common Errors

(i) The numbers 2, 3, 5, etc. are not fractions.

(ii) $\frac{8}{8}$ is not an improper fraction.

(iii) $\frac{15}{8} > \frac{15}{7}$, because 8 > 7.

(iv) Mixed fractions and improper fractions are the same.

(v) $5\frac{3}{7}$ and $3\frac{5}{7}$ are the same

(vi) $\frac{15}{4}$ is a mixed fraction

(vii) $\frac{2}{3}+\frac{5}{8}=\frac{2+5}{3+8}=\frac{7}{11}, \frac{2}{5}+\frac{3}{5}=\frac{5}{10}$

(viii) $\frac{2}{3}-\frac{5}{8}=\frac{2-5}{3-8}=\frac{-3}{-5}=\frac{3}{5}$

(ix) $\frac{2}{3}\div\frac{5}{4}=\frac{2\div5}{3\div4}$

(x) $\frac{3}{8}=\frac{3\times0}{8\times0}$

(xi) Equivalent fraction of $\frac{3}{5}$ with numerator 6 is $\frac{6}{5}$.

(xii) Equivalent fraction of $\frac{3}{5}$ is $\frac{3+4}{5+4}$

(xiii) $\frac{16}{64}=\frac{1}{4}$ because $\frac{1\not{6}}{\not{6}4}=\frac{1}{4}$ (cancelling 6 from numerator and denominator)

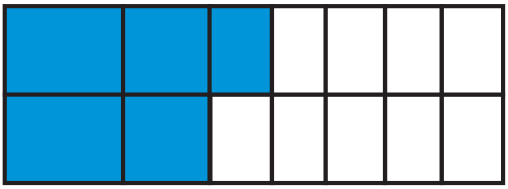
(xiv) Simplest form of $\frac{16}{28}$ is $\frac{8}{14}$

(xv) Equivalent fraction of $\frac{4}{9}$ with denominator 63 is $\frac{4}{63}$.

(xvi) Equivalent fraction of $\frac{20}{35}$ with denominator 7 is $\frac{20}{7}$

(xvii) $\frac{3}{8}, \frac{3}{9}, \frac{3}{5}, \frac{3}{4}$ are like fractions.

(xviii)



Shaded portion represents $\frac{5}{14}$

(xix) In a fraction, denominator is always greater than the numerator.

(xx) 0.3 + 0.03 = 0.06 or 0.3 + 0.03 = 0.6

(xxi) 0.4 × 4 = 0.16

(xxii) Sometimes student interpret term 'One-half' as

which is incorrect, (they should be told that.

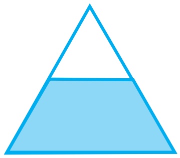
'$1\frac{1}{2}$' is one and half where as one-half is $\frac{1}{2}$).

Through the following exercise you may evaluate students.

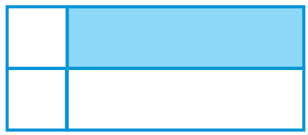
Exercise

1. What fraction of a day is 8 hours?

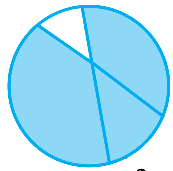
2. Identify the error in the following, if any.



This is $\frac{1}{2}$



This is $\frac{1}{4}$



This is $\frac{3}{4}$

3. Fill in the blanks using >, < or =

(i) $\frac{3}{5}$ ..... $\frac{3}{8}$ (ii) $\frac{12}{7}$ ..... $\frac{18}{14}$ (iii) $\frac{6}{9}$ ..... $\frac{12}{18}$

4. Express $\frac{37}{5}$ as a mixed fraction.

5. Express $6\frac{2}{5}$ as an improper fraction.

6. Write two fractions equivalent to $\frac{3}{7}$.

7. Express $\frac{38}{95}$ in the simplest form.

8. Find the sum of $\frac{12}{35}$ and $\frac{3}{14}$.

9. Find the value of $\frac{16}{45} - \frac{9}{10}$.

10. Find the product $\frac{6}{35} \times \frac{8}{9} \times \frac{7}{24}$.

11. Find the value of $\frac{14}{15} \div \frac{7}{10}$.

12. Find the value of 25.65 + 9.005 + 3.7

13. Find the value of 11.6 – 9.847

Unit 5

Rational Numbers

Structure

- Introduction
- Main Concepts and Sub-concepts
- Objectives
- Teaching Points
  1. Concept of rational numbers
  2. Positive and negative rational numbers
  3. Equivalent rational numbers
  4. Standard form of rational numbers
  5. Ordering of rational numbers
  6. Operations on rational numbers
  7. Properties of various operations on rational numbers
  8. Rational numbers between two rational numbers
- Common Errors
- Exercise

Introduction

The word 'Rational' is derived from the word 'ratio'. So, a rational number is defined as the ratio of two integers. The concept of ratio came into existence when we compared two quantities like masses, lengths, number of items, etc. But rational numbers have some similarity with the representation of ratio as division or fractions. Therefore, the topic of rational numbers is taught to the students after a fairly good knowledge of fractions.

At this stage, students should have enough practice of performing operations of addition, subtraction and multiplication on integers. They are now slowly moving away from concrete presentation of numbers to their abstract nature, because negative numbers cannot be presented as counting objects. Therefore, it is important to conduct activities on number line for visual presentation of integers.

Create situations in the classroom where students should be able to appreciate that dividing an integer by a non-zero integer not always results into an integer. Let the students think about such numbers, which are different from integers. Provide opportunities to the students to relate division of integers with division of natural numbers. Let them recall that division of natural number is also visualised as fractions. Therefore, the division of integers may have some similarity with fractions.

This new type of number of the form of an integer ÷ a non-zero integer or is called a rational number.

Keeping in view the similarity of fractions and rational numbers in representation and their operations, this topic has been introduced at the level of Class VII, after the chapter of fraction. Therefore, before introducing rational numbers in the class, ensure that learners have a proper understanding of fractions. Adequate practice of performing operations of addition, multiplication, subtraction and division on fractions helps in performing these operations on rational numbers as well.

Some common words like numerator, denominator, equivalent fractions, etc. have direct use in rational numbers. Therefore, equivalent rational numbers are defined as we do for equivalent fractions.

A brief idea of rational numbers, equivalent rational numbers and operations on rational numbers is given in this unit.

Main Concepts and Sub-concepts

- Rational numbers
- Equivalent rational numbers
- Simplest or standard form
- Operations on rational numbers.
- Properties of operations on rational numbers
- Rational numbers between two rational numbers

Objectives

After teaching this unit, the students will be able to

- feel the need for rational numbers;
- define rational numbers;
- write a rational number in standard form;
- compare two rational numbers as smaller than, bigger than or equal to;
- find sum, difference, product and quotient of two rational numbers;
- describe and verify properties of rational numbers on various operations like addition, subtraction, multiplication and division; and
- find rational numbers between two rational numbers.

Teaching Points

1. Concept of Rational Numbers

Students have already been taught about the natural numbers, whole numbers, integers and fractions.

They know that the differences of the form (5–7) are not defined in whole numbers. Therefore, the negative numbers alongwith the whole numbers form a new system called the system of integers. They also know that division of the form $5 \div 6$ is expressed as $\frac{5}{6}$ and this number is called a fraction. But what is the division of the form $-5 \div 6$ or $5 \div (-6)$? Clearly the quotient in each case is neither an integer nor a fraction.

In other words, we can say that when we divide an integer by another non-zero integer the result may not be an integer. Now the question arises, what this number is? In order to give answer to this question, a new form of numbers is defined. These are called rational numbers. Therefore, $-5 \div 6$ is a rational number and can be expressed as $-\frac{5}{6}$. Similarly, $5 \div (-6)$ is also a rational number which can be expressed as $\frac{5}{-6}$. Let us learn more about rational numbers.

DEFINITION OF A RATIONAL NUMBER

When an integer is divided by another non-zero integer, the quotient is referred to as rational number. For example $\frac{2}{3}$, 6, 5, –3, $-\frac{1}{5}$, etc. are all rational numbers. Did you notice that we wrote 6, 5, –3 as rational numbers? Can you tell why we call these numbers, in fact all integers, as rational numbers?

Because, any integer may be taken as the quotient (result) of dividing the integer by 1, i.e. $6 = \frac{6}{1}$, $5 = \frac{5}{1}$, $-3 = \frac{-3}{1}$ etc.

So, we conclude that all integers are also rational numbers.

Form groups of three or four children each and let them discuss whether all fractions, are rational numbers. Also let them find whether all rational numbers are fractions.

Clearly a rational number of the form $\frac{-3}{4}$ is not a fraction.

Mathematically, we define a rational number as a number which can be put in the form $\frac{p}{q}$ where p and q are integers and q is non-zero integer. Here p is called the numerator and q is the denominator of the rational number $\frac{p}{q}$. Encourage students to explore that $\frac{a}{0}$, $a \neq 0$ is not a rational number, because division by 0 is not defined. At this stage, it is also interesting for students to note that a number of the form $2\frac{1}{3}$ is also rational number as it can be written as $\frac{7}{3}$.

2. Positive and Negative Rational Numbers

Let the student recall that in integers it was studied that

positive integer ÷ positive integer = positive integer
positive integer ÷ negative integer = negative integer
negative integer ÷ positive integer = negative integer and
negative integer ÷ negative integer = positive integer

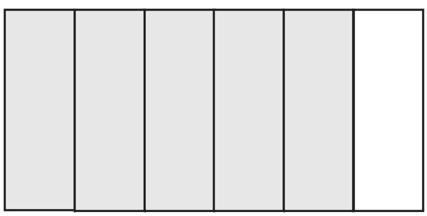
By using these facts, we can say that if both p and q have like signs, i.e. both are either positive or negative, the number $\frac{p}{q}$ will be **positive**. For example, $\frac{6}{7}, \frac{16}{18}, \frac{-100}{-77}, \frac{-5}{-1}$ are all positive rational numbers. If p and q have opposite signs, i.e. one is positive and the other is negative, the number $\frac{p}{q}$ will be negative.

For example, $\frac{6}{-7}, \frac{-17}{8}, \frac{-100}{10}$ are all negative rational numbers.

Students may note that zero is also a rational number, since we can write zero as = $\frac{0}{1}$, which is the quotient of two integers with a non-zero denominator.

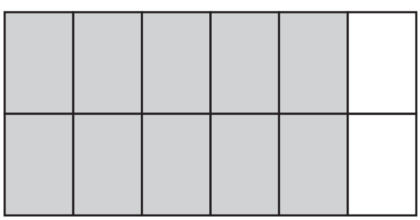
3. EQUIVALENT RATIONAL NUMBERS

Recall that $\frac{5}{6}$ and $\frac{10}{12}$ are called equivalent fractions, because they both represent the same part of a whole as shown in following figures 5.1(i) and 5.1(ii) by shaded parts.



(i)

Shaded part represents $\frac{5}{6}$ of the whole figure



(ii)

Shaded part represents $\frac{10}{12}$ of the whole figure

Fig 5.1

We also know that $\frac{10}{12}$ can be obtained by multiplying the numerator and denominator of $\frac{5}{6}$ by 2. Also $\frac{5}{6}$ can be obtained from $\frac{10}{12}$ by dividing the numerator and denominator of $\frac{10}{12}$ by 2.

So, equivalent fractions can be obtained by multiplying or dividing the numerator and denominator of the given fraction by a non-zero integer. Like fractions, we can also define equivalent rational numbers as follows:

If m is a non-zero integer and $\frac{p}{q}$ is a given rational number, then $\frac{p \times m}{q \times m}$ is a rational number equivalent to $\frac{p}{q}$.

For example, $\frac{-1}{2}$ and $\frac{-1 \times 5}{2 \times 5} = \frac{-5}{10}$ are equivalent rational numbers.

Similarly, $\frac{3}{-5}$ and $\frac{3 \times -1}{-5 \times -1} = \frac{-3}{5}$ are also equivalent rational numbers. From the above example, we find that we can always write a given rational number in its equivalent form with positive denominator (if it is negative).

You are aware that drill and practice has an important place in learning mathematics. For each concept, teacher should give opportunity to the students to solve some problems based on that concept and the algorithm. Few questions are given in the textbook in the form of exercises. But encourage students to form their own problems. You can also make many new problems for students. Some examples are given below.

Write each of the following as an equivalent rational number with positive denominator.

(a) $\frac{-4}{6}$ (b) $\frac{-5}{15}$ (c) $\frac{1}{-10}$

Observe the following pairs of rational numbers.

(a) $\frac{-6}{4}$ and $\frac{-3}{2}$ (b) $\frac{18}{-24}$ and $\frac{3}{-4}$

Do you observe any relation between the numbers in each pair? Yes, we can obtain the second number by dividing the numerator and the denominator of the first number by the same number. So, we say that $\frac{-6}{4}$, $\frac{-3}{2}$ are equivalent rational numbers.

Also $\frac{18}{-24}$ and $\frac{3}{-4}$ are equivalent.

In general, if $\frac{p}{q}$ is a rational number, then $\frac{p \div n}{q \div n}$ is an equivalent rational number, where n is a non-zero integer.

For example:

$$\frac{-12}{18}=\frac{-12\div 2}{18\div 2}=\frac{-6}{9}$$

$$\frac{-12}{18}=\frac{-12\div 3}{18\div 3}=\frac{-4}{6}$$

$$\frac{-12}{18}=\frac{-12\div 6}{18\div 6}=\frac{-2}{3}$$

i.e. $\frac{-12}{18}, \frac{-6}{9}, \frac{-4}{6}, \frac{-2}{3}$ are all equivalent rational numbers.

Therefore, by multiplying or dividing the numerator and the denominator of a given rational number by the same non-zero number, we can find as many equivalent rational numbers as we want. In fact, there lie infinitely many rational numbers equivalent to a given rational number. An example of writing desired number of equivalent rational numbers is given below.

Example: Write three equivalent rational numbers to $\frac{16}{-12}$.

Solution:

$$\frac{16}{-12}=\frac{16\div 2}{-12\div 2}=\frac{8}{-6}$$

$$\frac{16}{-12}=\frac{16\div 4}{-12\div 4}=\frac{4}{-3}$$

$$\frac{16}{-12}=\frac{16\div(-4)}{-12\div(-4)}=\frac{-4}{3}$$

So, $\frac{8}{-6}, \frac{4}{-3}$ and $\frac{-4}{3}$ are the three equivalent rational numbers to $\frac{16}{-12}$.

4. STANDARD FORM OF RATIONAL NUMBERS

Observe the following rational numbers:

$$\frac{-3}{4}, \frac{-1}{5}, \frac{2}{7}, \frac{-5}{2}$$

In all of the above rational numbers, the denominators are all positive. Also there is no common factor (other than 1) in the numerator and denominator of each of the numbers.

A rational number in such form is said to be written in the standard or simplest form. **A rational number is said to be in standard or simplest form when its numerator and denominator have no common factors other than 1 and the denominator is positive.**

Example: Express each of the following in simplest (standard) form.

(a) $\frac{14}{-70}$ (b) $\frac{-552}{-216}$

Solution: (a) $\frac{-14}{70} = \frac{14 \div 7}{-70 \div 7} = \frac{2}{-10}$

$$\frac{2}{-10} = \frac{2 \div 2}{-10 \div 2} = \frac{1}{-5}$$

$$\frac{1}{-5} = \frac{1 \times -1}{-5 \times -1} = \frac{-1}{5}$$

So, the standard or simplest form of $\frac{14}{-70}$ is $\frac{-1}{5}$

(b) $\frac{-552}{-216} = \frac{-552 \times -1}{-216 \times -1} = \frac{552}{216} = \frac{552 \div 2}{216 \div 2} = \frac{276}{108}$

$$\frac{276}{108}=\frac{276 \div 2}{108 \div 2}=\frac{138}{54}$$

$$\frac{138}{54}=\frac{138 \div 2}{54 \div 2}=\frac{69}{27}$$

$$\frac{69}{27}=\frac{69 \div 3}{27 \div 3}=\frac{23}{9}$$

So, the simplest form of $\frac{-552}{-216}$ is $\frac{23}{9}$

Short cut method

In the above examples we obtained the equivalent rational numbers by repeatedly dividing the numerator and denominator by common factors.

This process can be shortend by dividing the numerator and denominator by their HCF, i.e. highest common factor of their absolute values.

For example, HCF of 14 and 70 = 14

$$\frac{-14}{70}=\frac{14 \times -1}{-70 \times -1}=\frac{-14}{70}$$

and $$\frac{-14}{70}=-\frac{14 \div 14}{70 \div 14}=-\frac{1}{5}$$

Similarly $\frac{-552}{-216}=\frac{-552 \times -1}{-216 \times -1}=\frac{552}{216}$

HCF of 552 and 216 = 24

Therefore, $\frac{-552}{-216}=\frac{-552 \div 24}{-216 \div 24}=\frac{-23}{-9}=\frac{23}{9}$

Working steps to write simplest form of a given rational number:

Step 1 Make the denominator +ve (if not) by multiplying the numerator and denominator by – 1.

Step 2 Find HCF of the numerator and the denominator by ignoring their sign (if – ve).

Step 3 Divide numerator and denominator of the number obtained in Step 1 by the HCF obtained in Step 2.

The result will be the simplest form of the given rational number.

Methods to verify that given pairs of rational numbers are equivalent or not

We have already studied that two rational numbers are equivalent if one of them can be obtained from the other by multiplying or dividing the numerator and the denominator by the same non-zero integer.

So, this may be a method to verify that two rational numbers are equivalent or not.

For example $\frac{-4}{9}$ and $\frac{12}{-27}$ are equivalent because $\frac{12}{-27}$ can be obtained from $\frac{-4}{9}$ by multiplying the numerator and the denominator by – 3.

Now can you tell whether $\frac{18}{-27}$ and $\frac{4}{-6}$ are equivalent? In this case, rational number cannot be obtained by multiplying or dividing numerator and denominator by the same number. There is another method for checking this. Students have already studied about this method in fractions, called the method of cross multiplication.

Recall that in this method; if the product of the numerator of the first number by the denominator of the second and the product of the denominator of the first number by the numerator of the second are equal, we say that the two fractions are equivalent. Same is also true in the case of rational numbers.

Let us now use the cross multiplication method for $\frac{18}{-27}$ and $\frac{4}{-6}$

$$\frac{18}{-27} \times \frac{4}{-6}$$

$18 \times (-6) = -108$, $4 \times (-27) = -108$

That is, $18 \times (-6) = 4 \times (-27) = -108$

Therefore, $\frac{18}{-27}$ and $\frac{4}{-6}$ are equivalent.

In general, two rational numbers $\frac{p}{q}$ and $\frac{a}{b}$ are equivalent if $p \times b = q \times a$.

Example: Determine the number which can be put in the blank box.

(a) $\frac{-2}{7} = \frac{\square}{42}$ (b) $\frac{5}{-9} = \frac{-10}{\square}$

Solution: (a) $\frac{-2}{7} = \frac{\square}{42}$ will be equivalent if -2×42 is equal to $\square \times 7$.

or $\square \times 7 = -84$.

So $\square$ can be filled by the number which when multiplies 7 gives –84.

We can also write it as $\square = \frac{-84}{7} = -12$

(b) $\frac{5}{-9} = \frac{-10}{\square}$ will be equivalent, if $5 \times \square$ is equal to $-9 \times (-10) = 90$.

So $5 \times \square = 90$ $\square = 90 \div 5 = 18$

Hence, the required number in the box is 18.

Some simple problems based on the above method can be made by the teachers in addition to the exercises given in the book as given below:

1. Write the numerator of each of the following rational numbers

   (a) $\frac{-4}{5}$ (b) $\frac{4}{-5}$ (c) $\frac{1}{5}$ (d) -7

2. Write the denominator of each of the following rational numbers

   (a) $\frac{-5}{7}$ (b) $\frac{7}{-5}$ (c) $\frac{9}{-15}$ (d) 10

3. Write rational numbers in simplest form whose numerators and denominators are given below

   (a) Numerator = 4, denominator = – 3

   (b) Numerator = – 18, denominator = – 36

   (c) Numerator = 8 × 7, denominator = 7 – 8

   (d) Numerator = 17 + 5, denominator = (– 2) × (– 11)

5. ORDERING OF RATIONAL NUMBERS

Two rational numbers can be compared on the basis of the comparison of integers and of fractions. As numbers, we now may say that rational numbers are signed fractions, (fraction with negative (–) or positive (+) sign). Students already know how to compare the two integers.

Every positive integer is greater than every negative integer and zero.

Every negative integer is smaller than every positive integer and zero.

Keeping this analogy in mind, two rational numbers with opposite signs are compared, that is,

Every positive rational number is greater than all negative rational numbers and zero.

Every negative rational number is smaller than all positive rational numbers and zero.

In order to compare two rational numbers with the same sign (i.e., both positive or both negative) we use the concepts of comparing integers with same sign and the concept of comparing two fractions. For example to compare $-\frac{4}{5}$ and $-\frac{6}{7}$ we ignore their sign and compare $\frac{4}{5}$ and $\frac{6}{7}$. We can write $\frac{4}{5}=\frac{4\times7}{5\times7}=\frac{28}{35}$ and $\frac{6}{7}=\frac{6\times5}{7\times5}=\frac{30}{35}$.

$$\text{Now } \frac{28}{35}<\frac{30}{35} \text{ i.e. } \frac{4}{5}<\frac{6}{7}.$$

Thus, $-\frac{4}{5}>-\frac{6}{7}$, because in integers students have studied that multiplying both sides of an inequality by a negative number reverses the inequality sign.

At this stage, it will be appropriate to ask students to develop their own algorithms, like in ordering of rational numbers first follow the steps as done for comparing two integers with same sign and then combine the steps of comparing two fractions after this.

6. OPERATIONS ON RATIONAL NUMBERS

Here again to add or subtract two rational numbers, methods of adding and subtracting two integers and the methods of adding and subtracting two fractions are used.

Addition of integers	**Addition of rational numbers**
When signs are same	
* Drop the sign and add	* Drop the sign and add the corresponding fractions.
* Write the common sign of the numbers with the result	* Write the common sign of the two rational numbers with the result. **Example:** $\frac{-5}{6}+\frac{-2}{3}=-\left[\frac{5+4}{6}\right]=\frac{-9}{6}$

When signs are opposite

Integers	Fractions
* Drop the signs and compare the two integers	* Drop the signs and compare the two fractions.
* Subtract the corresponding smaller number from the bigger number	* Subtract the corresponding smaller fraction from the bigger fraction.
* Assign the sign of the bigger number with the result.	* Assign the sign of the bigger fraction with the result

Note that the number obtained by dropping the negative sign of an integer (if present there) is called the absolute value of the integer. Therefore, absolute value of an integer is always non-negative.

But, we may do the operations without mentioning the concept of absolute value of an integer.

Example: $\frac{-5}{6}+\frac{2}{3}$

$\frac{2}{3}=\frac{4}{6}$

and, $\frac{5}{6}>\frac{4}{6}\Rightarrow\frac{5}{6}>\frac{2}{3}$

Also $\frac{5}{6}-\frac{4}{6}=\frac{5-4}{6}=\frac{1}{6}$

The sign of the bigger fraction

i.e. $\frac{5}{6}$ is –ve.

Therefore, $\frac{-5}{6}+\frac{2}{3}=\frac{-1}{6}$

Example: $\frac{7}{8}+\left(\frac{-11}{6}\right)$

Now,

$\frac{7\times3}{8\times3}=\frac{21}{24},\ \frac{11\times4}{6\times4}=\frac{44}{24}$

Clearly $\frac{21}{24}<\frac{44}{24}$ i.e. $\frac{7}{8}<\frac{11}{6}$

Now $\frac{11}{6}-\frac{7}{8}=\frac{44}{24}-\frac{21}{24}=\frac{23}{24}$

The sign of the bigger fraction

i.e. $\frac{11}{6}$ is –ve.

Therefore, $\frac{7}{8}+\frac{-11}{6}=\frac{-23}{24}$

Same way the operations of subtraction, multiplication and division on rational numbers are performed. First the signs are handled as we do for integers then the respective operation is applied to corresponding fractions.
Some examples are given below:

$\frac{-5}{6}-\left(\frac{-7}{8}\right)=-\frac{5}{6}+\frac{7}{8}$ (because subtracting a number means adding its additive inverse, i.e. its corresponding negative)

$=\frac{-20+21}{24}=\frac{1}{24}$

$-\frac{5}{6}\times\frac{-7}{8}=\frac{5}{6}\times\frac{7}{8}=\frac{35}{48}$ (because negative multiplied by negative gives +ve product)

Similarly, $-\frac{5}{6}\times\frac{7}{8}=-\left(\frac{5}{6}\times\frac{7}{8}\right)$ (because negative multiplied by positive gives –ve product)

$=-\frac{35}{48}$

$\frac{-5}{6}\div\left(-\frac{7}{8}\right)=\frac{5}{6}\div\frac{7}{8}$ (because negative divided by negative gives +ve quotient)

$$= \frac{5}{6} \times \frac{8}{7} = \frac{20}{21}$$

(meaning of division by a fraction is multiplying by its reciprocal).

$$-\frac{5}{6} \div \frac{7}{8} = -\left(\frac{5}{6} \div \frac{7}{8}\right)$$

(because negative divided by positive gives negative quotient)

$$= -\left(\frac{5}{6} \times \frac{8}{7}\right) = -\frac{40}{42} = \frac{-20}{21}$$

7. PROPERTIES OF VARIOUS OPERATIONS ON RATIONAL NUMBERS

The students are now expected to have appropriate understanding of properties of operations on integers, like **closure**, **commutative**, **associative**, **existence of additive identity**, **existence of additive inverse and existence of multiplicative identity**. They also know that multiplication of integers distributes over addition and subtraction. Involve students in activities to verify the above properties for rational numbers also. Encourage them to realise that in addition to these properties, rational numbers have few more properties which are not satisfied by integers.

One such property is the existence of multiplicative inverse of a non-zero rational number, i.e. for any non-zero rational number, there lies another rational number which when multiplied by the given number gives the product 1,i.e. multiplicative identity.

The teacher may explain by taking a rational number, say $\frac{-4}{9}$. Then its multiplicative inverse will be $\frac{9}{-4}$ or $\frac{-9}{4}$ because $\frac{-4}{9} \times \frac{-9}{4} = 1$. By taking other such examples it should be made clear to the students that multiplicative inverse of rational number $\frac{p}{q}$ is $\frac{q}{p}$ provided $p \neq 0$. Thus, every non-zero rational number has its multiplicative inverse. The multiplicative inverse can also be called the reciprocal of the given rational number.

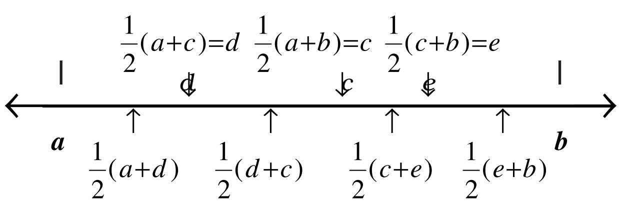
The teacher may also show to the students that like fractions all the rational numbers can be represented or located on the number line.

8. RATIONAL NUMBERS BETWEEN TWO RATIONAL NUMBERS

Students know that between any two consecutive integers there lies no integer. They also know about the representation of fractions and integers on a number line. Rational numbers can be represented in a similar way. All positive rational numbers are represented as points on a number line to the right of the point corresponding to number 0, called the origin. All negative numbers are represented to the left of the origin on the number line.

Give enough practice to the students to make them realise that corresponding to every rational number there lies a point on the number line. Also let them realise that half of the sum of any two rational numbers lies between them on the number line. So if a and b are any two rational numbers, we find that

lies between a and b. Same way half of the sum of a and $\frac{1}{2}(a+b)$ must lie between a and b, and also half the sum of $\frac{1}{2}(a+b)$ and b must lie between a and b. The process will go on for indefinitely many times as shown in following figure:



Can we conclude that between any two rational numbers there lie infinitely many rational numbers?

Thus, we can introduce as many rational numbers between two given rational numbers, as we like. One of the methods is given above. This method is called **method of average** because $\frac{a+b}{2}$ is the **average of** a **and** b.

We may also introduce any desired number of rational numbers between two given rational numbers by another method. For this, the equivalent rational numbers of both the given numbers are obtained with same denominators. Then by using the ordering of rational numbers, the desired number of rational numbers can be found between them.

The method is demonstrated below:

Example: To introduce rational numbers between $\frac{1}{3}$ and $\frac{1}{2}$

We know that $\frac{1\times 20}{3\times 20}=\frac{20}{60}$ and $\frac{1}{2}=\frac{1\times 30}{2\times 30}=\frac{30}{60}$

$$\frac{20}{60}<\frac{21}{60}<\frac{22}{60}<\frac{23}{60}<\frac{24}{60}\ldots\ldots\ldots\ldots\frac{29}{60}<\frac{30}{60}$$

$$=\frac{1}{3}<\frac{21}{60}<\frac{22}{60}<\frac{23}{60}<\ldots\ldots\ldots\ldots\frac{29}{60}<\frac{1}{2}$$

So, $\frac{21}{60},\frac{22}{60},\ldots\ldots\frac{29}{60}$ are 9 rational numbers between $\frac{1}{3}$ and $\frac{1}{2}$.

In case we have to introduce more number of rational numbers we may write the equivalent rational numbers as $\frac{1\times 60}{2\times 60}=\frac{60}{120}$ and $\frac{1\times 40}{3\times 40}=\frac{40}{120}$ and then $\frac{41}{120},\frac{42}{120},\ldots\ldots\frac{59}{120}$ will lie between $\frac{1}{3}$ and $\frac{1}{2}$.

The property of existence of any number of rational numbers between two given rational numbers is also called the **denseness property** of rational numbers.
This property says that between any two rational numbers there lie infinitely many rational numbers, however close the given rational numbers may be.

Also on number line, students learnt that for every rational number we can find a point on the number line. But, in higher classes students will learn that there are many points on a number line which do not correspond to a rational number. This leads to a new class of numbers called *irrational numbers*. The detailed study of irrational numbers and the system of real numbers (rational numbers and irrational numbers) will be studied by students in higher classes.

Common Errors

(i) Every rational number is a fraction.

(ii) $\frac{2}{0}$ is a rational number since 0 is an integer.

(iii) $\frac{2}{-3}$ is not a rational number.

(iv) $4\frac{2}{3}$ is not a rational number.

(v) –1, 3, 4, 5 are not rational numbers.

(vi) 0 is not a rational number.

(vii) $\frac{2}{-3}$ and $\frac{-2}{3}$ are different rational number.

(viii) $-\frac{1}{6} > -\frac{1}{8}$

(ix) Reciprocal of the rational number $\frac{3}{2}$ is $\frac{-2}{3}$

(x) 0 is a negative and also a positive rational number.

(xi) $\frac{-1}{-2}$ is a negative rational number.

(xii) Reciprocal of the rational number $\frac{-4}{9}$ is $\frac{9}{4}$.

(xiii) $\frac{4}{11} \div 0 = 0$

(xiv) Between two rational numbers, say and $\frac{2}{9}$, there is no rational number.

(xv) Between two rational numbers, $\frac{1}{8}$ and $\frac{2}{3}$ there is only one rational number $\frac{\frac{1}{8}+\frac{2}{3}}{2}$.

(xvi) (a) $\frac{2}{5}-\frac{-2}{5}=\frac{0}{5}=0$

(b) $\frac{2}{5}+\frac{2}{-5}=\frac{4}{5-5}=\frac{4}{0}$

(xvii) Additive inverse of $-\frac{7}{3}$ is $\frac{3}{-7}$

(xviii) A rational number is positive only when its numerator and denominator both are positive and not when its numerator and denominator both are negative.

Teacher can evaluate the students through the following exercise.

Exercise

1. Find the sum:

   (i) $\frac{5}{4}+\left(\frac{-11}{4}\right)$ (ii) $\frac{5}{3}+\frac{3}{5}$ (iii) $\frac{-9}{10}+\frac{22}{15}$

2. Find

   (i) $\frac{7}{24}-\frac{17}{36}$ (ii) $\frac{5}{63}+\left(\frac{-6}{21}\right)$ (iii) $\frac{-6}{13}-\left(\frac{-7}{15}\right)$

3. Find the product:

   (i) $\frac{9}{2}\times\left(\frac{-7}{4}\right)$ (ii) $\frac{3}{10}\times(-9)$ (iii) $\frac{-6}{5}\times\frac{9}{11}$

4. Find the value of:

   (i) $(-4)\div\frac{2}{3}$ (ii) $\frac{-3}{5}\div 2$ (iii) $\frac{-4}{5}\div(-3)$

5. Represent these numbers on the number line

   (i) $\frac{7}{4}$ (ii) $\frac{-5}{6}$

6. Represent $\frac{-2}{11},\frac{-5}{11},\frac{-9}{11}$ on the number line.

7. Write five rational numbers which are smaller than 2.

8. Find ten rational numbers between $\frac{-2}{5}$ and $\frac{1}{2}$.

9. Find five rational numbers between

   (i) $\frac{2}{3}$ and $\frac{4}{5}$ (ii) $\frac{-3}{2}$ and $\frac{5}{3}$ (iii) $\frac{1}{4}$ and $\frac{1}{2}$

10. Write five rational numbers greater than –2.

Unit 6

Squares and Square Roots

Structure

- Introduction
- Main Concepts and Sub-concepts
- Objectives
- Teaching Points
  1. Squares
  2. Properties of square numbers
  3. Patterns of some square numbers
  4. Finding square of a number
  5. Pythagorean triplets
  6. Finding square roots
     - 6.1 Square root through repeated subtraction
     - 6.2 Square root through prime factorisation
     - 6.3 Square root by division method
  7. Finding square roots of decimal numbers
  8. Estimating a square root
- Common Errors
- Exercise

Introduction

The students already know the multiplication of two numbers, such as

$$2 \times 2 = 4, \quad 2 \times 3 = 6, \quad 2 \times 4 = 8$$
$$3 \times 2 = 6, \quad 3 \times 3 = 9\,, \quad 3 \times 4 = 12, \text{ etc.}$$

When a number is multiplied once with itself the product so obtained is called the **square** of the number. The number itself is called the **square root** of the square so obtained. Thus, the numbers 4 and 9 are the **squares** of the numbers 2 and 3, respectively and the numbers 2 and 3 are the **square roots** of the numbers 4 and 9, respectively.

In this unit, we will discuss squares and square roots of numbers.

Main Concepts and Sub-concepts

- Square numbers and their properties
- Squares of two and three digit numbers
- Square roots of perfect square numbers through prime factorisation method and division method
- Square roots of decimal numbers

Objectives

After teaching this unit the students will be able to

- understand the meaning of a square number;
- distinguish between a square number and a non-square number;
- know some properties and patterns of square numbers;
- understand the meaning of square root of a number;
- find square root of a number through (i) prime factorisation (ii) division method; and
- find square root of a decimal number.

Teaching Points

1. SQUARES

You may introduce the concept of a square number by asking the students to recall the area of a square with side of length 'a' units

$$\text{Area} = a \times a = a^2$$

Ask them to complete the following table:

Table 1

Side of a square (in cm)	Area (in cm^2)
1	$1 \times 1 = 1^2 = 1$
2	____________
3	$3 \times 3 = 3^2 = 9$
4	____________
5	____________
6	$6 \times 6 = 6^2 = 36$
7	____________
8	____________
9	____________
10	$10 \times 10 = 10^2 = 100$

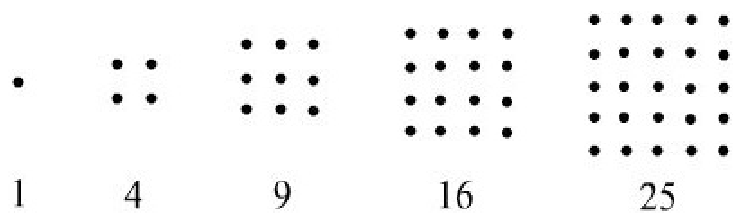
Numbers like $4 = 2 \times 2$, $9 = 3 \times 3$, $16 = 4 \times 4$, $25 = 5 \times 5$, $36 = 6 \times 6$, $49 = 7 \times 7$, $64 = 8 \times 8$, $81 = 9 \times 9$, $100 = 10 \times 10$, etc. are the numbers which can be expressed as the product of a number by itself.

Such numbers as 1, 4, 9, 16, ... , 100 are known as **square numbers.**

In general, if a natural number is a square of some natural number, it is called a perfect square or a square number.

In other words, a natural number n is a perfect square or a square number if $n = m^2$, for some natural number m.

Let the students notice that square numbers 1, 4, 9, 16, ---, etc. can be represented geometrically by dots forming a square as shown below:



Hence the name **square numbers is justified.**

How to know whether a given number is a square number or not?

Write the number as a product of its prime factors. If the number is a square number, then you would be able to make different pairs such that both factors of each pair will be the same.

Example: Check whether 225 is a square number or not.

Solution: 225 = **3** × **3** × **5** × **5** [prime factorisation]

Prime factors on the right are paired in such a way that both the factors in each pair are the same.

So, 225 is a square number.

Note that it is a square of 3 × 5 = 15.

Check: 15 × 15 = 225.

Example: Is 32 a square number?

Solution: $32 = 2 \times 2 \times 2 \times 2 \times 2$

$= 2 \times 2 \quad \times \quad 2 \times 2 \quad \times \quad 2$

Prime factors of 32 when grouped into pairs of same factors, we see that one factor 2 is left which is not paired. So, 32 is not a square number.

1. Ask the students to find whether the following numbers are square numbers or not.

   (i) 64 (ii) 36 (iii) 22 (iv) 252

2. Find the square numbers between 30 and 40

2. PROPERTIES OF SQUARE NUMBERS

(i) Let the students extend the Table 1 to the numbers 11, 12, 13,...., 20 and observe the digit in units' place of the square numbers.

Help them in finding that the square numbers end in **0, 1, 4, 5, 6 and 9 only**.

This suggests **that a number with 2, 3, 7 or 8 in its unit place is not a square number.**

Can we say that if a number ends in 0, 1, 4, 5, 6 or 9, then it must be a square number?

Answer is 'not always' Think about the number 19. About the number 50!

(ii) The number of zeroes at the end of a square numbers is always **even**. For example 100 is a square number. 2500 is also a square number. Ask the students to check (using prime factorisation) the statement by taking some square numbers having even number of zeroes at the end.

If a number has **odd** number of zeroes at the end, will it be a square number?

Check whether 250, 1000 are square numbers. (using prime factorisation).

(iii) See Table 1 with numbers and their squares. Let them observe that

(a) squares of even numbers are even.

(b) squares of odd numbers are odd.

(iv)

Units place of the number	Units place of its square
0	0
1	1
2	4
3	9
4	6
5	5
6	6
7	9
8	4
9	1
10	0

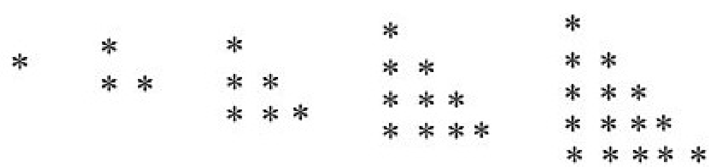
3. PATTERNS OF SOME SQUARE NUMBERS

(i) Adding triangular numbers

Triangular numbers are: 1, 3, 6, 10, 15, ...

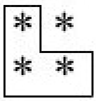
Why are they called triangular numbers?

They are called triangular numbers because dot patterns can be arranged as triangles.

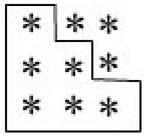


1 3 6 10 15

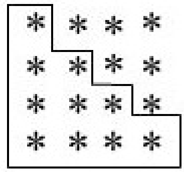
If two consecutive triangular numbers are added, we get a square number.



$1 + 3 = 4$
$= 2^2$



$3 + 6 = 9$
$= 3^2$



$6 + 10 = 16$
$= 4^2$

(ii) Numbers between square numbers

For two consecutive numbers n and $(n + 1)$,

$(n + 1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1$

This shows that between squares of two consecutive numbers, n^2 and $(n+1)^2$, there are $2n + 1 - 1 = 2n$ non-square numbers.

Teachers may help the students to verify it by taking different values of n. For example, if $n = 3$

$(3 + 1)^2 - 3^2 = 7 = 6 + 1$

i.e. there are 6 non square numbers between 9 and 16. They are
10, 11, 12, 13, 14, 15

6

(iii) Adding odd numbers

Sum of first n odd numbers is n^2

Teacher can verify this statement by taking first one, first two or first three odd numbers as follows :

1	$=$	1	$=$	1^2
$1+3$	$=$	4	$=$	2^2
$1 + 3 + 5$	$=$	9	$=$	3^2

The students may observe and extend the pattern further and so on.

(iv) If n is a square number, then $2n$ can never be a square number

This statement can be verified for square numbers given in Table 1.

25 is a square number, but $2 \times 25 = 50$ is not a square number

36 is a square number, but $2 \times 36 = 72$ is not a square number and so on.

(v) An odd square number can be expressed as the sum of two consecutive numbers

Verify this statement as

Odd square numbers				Sum of two consecutive numbers
3^2	=	9	=	4 + 5
5^2	=	25	=	12 + 13
7^2	=	49	=	24 + 25

(vi) Some more patterns in square numbers

(a)

$1^2 = 1$

$11^2 = 121$ and $1 + 2 + 1 = 2^2$

$111^2 = 12321$ and $1 + 2 + 3 + 2 + 1 = 3^2$

$1111^2 = 1234321$ and $1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$

and so on.

Ask the students to write the squares 11111^2 and 111111^2 using the above pattern.

(b) See the following pattern:

$11^2 = 121$

$101^2 = 10201$

$1001^2 = 1002001$

$10001^2 = 100020001$

. . .

. . .

Ask the students to write 100001^2, 1000001^2, 10000001^2, 100000001^2 using the above pattern, (not by actual multiplication).

(c) See the following pattern:

$$11^2 = 121$$
$$101^2 = 10201$$
$$10101^2 = 102030201$$
$$\vdots$$

Let the students write 1010101^2, 101010101^2, 10101010101^2 using the above pattern (not by actual multiplication).

(d) See the following pattern:

$1^2 + 2^2 + 2^2$	=	3^2			
			2	=	1×2
$2^2 + 3^2 + 6^2$	=	7^2	6	=	2×3
$3^2 + 4^2 + 12^2$	=	13^2	12	=	3×4
$4^2 + 5^2 + 20^2$	=	21^2	20	=	4 x 5

See how the third number (i.e. 2^2, 6^2 12^2, 20^2) is related to first two numbers.

Let the students find

$5^2 + 6^2 + 30^2$, $6^2 + 7^2 + 42^2$, $7^2 + 8^2 + 56^2$, etc. through the above pattern. Here again, the teacher may see that the students should not write them by actual multiplication.

4. FINDING SQUARE OF A NUMBER

(a) Students can find squares of small numbers such as 2, 3, 4, 5, 6, 7... easily by multiplying the number with itself.

How to find square of say 23 quickly?

One way is to multiply 23 by 23 as

$$\begin{array}{r} 23 \\ \times 23 \\ \hline 69 \\ 46\times \\ \hline 529 \\ \hline \end{array}$$

Another way is

$23 = 20 + 3$

$23^2 = (20+3)^2 = (20+3)\ (20+3)$

$= 20\ (20+3) + 3\ (20+3)$ Distributive property

$= 20^2 + 20 \times 3 + 3 \times 20 + 3^2$

$= 400 + 60 + 60 + 9 = 529$

Give some more numbers such as 42, 49, 55, etc. to the students and ask them to find their squares by following the procedure discussed above and not by actual multiplication

5. PYTHAGOREAN TRIPLETS

If there are three positive numbers a, b and c such that

$$a^2 + b^2 = c^2$$

then the collection of numbers a, b, c is known as *Pythagorean Triplet.*

For example, the numbers 3, 4, 5 form a Pythagorean triplet as $3^2 + 4^2 = 5^2$.

Similarly numbers 5, 12, 13 form a Pythagorean triplet as $5^2 + 12^2 = 13^2$.

Do the numbers 5 , 6 , 7 from a Pythagorean triplet ? Let the students check and come up with their answers.

For more such triplets, use the numbers $2m$, m^2-1, m^2+1 as they form various Pythagorean triplets for different values of $m > 1$ (Here m has to be a natural number). This is true because

$(2m)^2 + (m^2-1)^2 = (m^2 + 1)^2$ is an identity and hence holds for all values of m.

Ask the students to find some more Pythagorean triplets using this identity.

6. Finding Square Roots

Refer to Table 1. We see

$$1 = 1^2$$
$$4 = 2^2$$
$$9 = 3^2$$
$$16 = 4^2$$
$$100 = 10^2$$

1, 4, 9, 16, ..., 100 are (perfect) square numbers. 1 is a square of itself, 4 is a square of 2, 9 is a square of 3, 16 is a square of 4 and so on.

We also say that : 2 is a **square root** of 4

3 is a **square root** of 9

4 is a **square root** of 16

and so on. Thus,

x is a square root of a number n if $n = x \times x$

Also $1^2 = (-1) \times (-1)$ — 1 is also a square root of 1

$2^2 = (-2) \times (-2)$ — 2 is also a square root of 2^2

$3^2 = (-3) \times (-3)$ — 3 is also a square root of 3^2 and so on.

Thus, there are two integral square roots of a perfect square number.

Positive square root of a number is denoted by the symbol $\sqrt{}$. For example

$\sqrt{4} = 2$ not -2 , $\sqrt{9} = 3$ not -3 , $\sqrt{16} = 4$ not -4 and so on.

6.1 SQUARE ROOT THROUGH REPEATED SUBTRACTION

Ask the students to recollect that

Sum of first n odd numbers is n^2

Now you can apply this result to find square root of say 81, i.e. $\sqrt{81}$

We now subtract successive odd numbers from 81.

Starting from the subtraction of 1 from 81, we get.

Step 1 : 81 – 1 = 80

Step 2 : 80 – 3 = 77

Step 3 : 77 – 5 = 72

Step 4 : 72 – 7 = 65

Step 5 : 65 – 9 = 56

Step 6 : 56 – 11 = 45

Step 7 : 45 – 13 = 32

Step 8 : 32 – 15 = 17

Step 9 : 17 – 17 = 0

Note that 1, 3, 5, 7, 9, 11, 13, 15, 17 are successive first nine odd numbers and **you obtained 0 in the 9th step.** Therefore, square root of 81 is 9 or $\sqrt{81} = 9$.

Check : $9 \times 9 = 81$

Ask the children to find

(i) (ii) $\sqrt{49}$ (iii) $\sqrt{121}$

by using the method discussed above.

6.2 SQUARE ROOT THROUGH PRIME FACTORISATION

Ask the students to find the square root of, say 729, using the method of repeated subtraction. Let them do it and come up with the feeling that it is lengthy and time-consuming process.

So, there is need to look for a simpler way. Let us find prime factorisation of 729.

$$729 = \underbrace{3 \times 3 \times 3 \times 3 \times 3 \times 3}_{\text{Prime factorisation}}$$

Pair the prime factors as

$$\begin{aligned} 729 &= 3 \times 3 \quad \times \quad 3 \times 3 \quad \times \quad 3 \times 3 \\ &= 3^2 \quad \times \quad 3^2 \quad \times \quad 3^2 \\ &= (3 \times 3 \times 3)^2 \end{aligned}$$

So, $\sqrt{729} = 3 \times 3 \times 3 = 27$

Prime factorisation of a number means to express the given number as a product of prime numbers.
For example :
$6 = 2 \times 3$, $8 = 2 \times 2 \times 2$
$12 = 2 \times 2 \times 3$, $16 = 2 \times 2 \times 2 \times 2$
$36 = 2 \times 2 \times 3 \times 3$

Students may note that a perfect square number can always be expressed as the product of pairs of equal factors.

It may be summarised that to find the square root of a perfect square number we use the following steps:

(i) Form prime factors of the given number
(ii) Make pairs of similar prime factors.
(iii) Take the product of the prime factors, choosing one factor out of every pair.

Ask the children to find

(i) $\sqrt{144}$ (ii) $\sqrt{256}$ (iii) $\sqrt{625}$ (iv) $\sqrt{400}$

Some more examples:

1. Check whether 48 is a square number or not.
   $48 = 2 \times 2 \times 2 \times 2 \times 3 = \underline{2 \times 2} \times \underline{2 \times 2} \times \textcircled{3}$
   Since all the prime factors of 48 are **not in pairs,** therefore 48 is **not** a square number
2. By which smallest number should 48 be multiplied so that the product becomes a perfect square? Find the square root of this product.

$48 = \underline{2 \times 2} \times \underline{2 \times 2} \times \textcircled{3}$

If we multiply by 3, to complete the pair, the product 48 × 3, i.e. 144 will become a perfect square

$144 = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{3 \times 3} = (2 \times 2 \times 3)^2$

So, $\sqrt{144} = 2 \times 2 \times 3 = 12$

3. By which smallest number should 48 be divided so that the quotient is a perfect square?

Since $48 = 2 \times 2 \times 2 \times 2 \times \textcircled{3}$, therefore

So, if 48 is divided by $\textcircled{3}$ we get a perfect square number and $48 \div 3 = 16$

Thus, 3 is the smallest such number.

Ask the children to find the following through prime factorisation method.

(i) $\sqrt{6400}$ (ii) $\sqrt{7056}$

6.3 SQUARE ROOT BY DIVISION METHOD

When the numbers are large, even the method of finding square roots by prime factorisation becomes lengthy and time consuming. Sometimes it is difficult also.

In such cases, we use **Long division method**.

Let us illustrate this method through an example.

Example: Find the square root of 6561.

Solution:

Step 1: Place a bar over every pair of digits starting from the digit at ones place. ⟶ $\overline{65}\,\overline{61}$

Step 2: Think of the largest square less than or equal to the number under the extreme left bar. (The number under extreme left bar can be of 1 digit or 2 digits). Here number under the extreme left bar is 65. So the largest square less than or equal to 65 is 64 or 8^2

Step 3: Take the square root of 8^2. It is 8. Take this number as the divisor and the quotient for the number under the left most bar. Divide and get the remainder. Here remainder is 1.

	8
8	$\overline{65}\ \overline{61}$
	64
	1

Step 4: Bring down the number under the next bar (it is 61 here) to the right of the remainder. So the new dividend is 161.

	8
8	$\overline{65}\ \overline{61}$
	64
	1 61

Step 5: Double the quotient as it appears (It is 8 here) and enter it with a blank on the right for the next digit as the next possible divisor.

	8
8	$\overline{65}\ \overline{61}$
	64
16_	1 61

Step 6: Guess a largest possible digit to fill the blank which will also become the next digit in the quotient such that when the new divisor is multiplied to the new quotient, the product is less than or equal to the dividend.

	81
8	$\overline{65}\ \overline{61}$
	64
161	1 61
	1 61
	0

So, $\sqrt{6561} = 81$

Step 7: Repeat steps 5 and 6, till the number under the last bar has been dealt.

Example: Find the square root of 531441

Solution:

		729
	7	$\overline{53}\ \overline{14}\ \overline{41}$
		49
Double of 7 in the quotient →	142	4 14
		2 84
Double of 72 in the quotient →	1449	1 30 41
		1 30 41
		0

So, $\sqrt{531441} = 729$

Example: Evaluate $\sqrt{4401604}$.

Solution:

		2098
	2	$\overline{4}\ \overline{40}\ \overline{16}\ \overline{04}$
		4
Double of quotient 2 →	40	0 40
		00
Double of quotient 20 →	409	40 16
		36 81
Double of quotient 209 →	4188	33504
		33504
		0

So, $\sqrt{4401604} = 2098$

Example: Find the least number which must be subtracted from 194491 to make it a perfect square. Find the square root of the resulting numbers.

	441
4	$\overline{19}\ \overline{44}\ \overline{91}$
	16
84	3 44
	3 36
881	8 91
	8 81
	10 ⟵ Remainder

Thus, if **10 is subtracted** from the given number, the square root of the remaining number 194481 will be 441.

Example: Find the least number which must be added to 306452 to make it a perfect square.

Solution:

	553
5	$\overline{30}\ \overline{64}\ \overline{52}$
	25
105	5 64
	5 25
1103	39 52
	33 09

	554
5	$\overline{30}\ \overline{64}\ \overline{52}$
	25
105	5 64
	5 25
1104	39 52
	44 16

From the above, it is clear that the given number is greater than $(553)^2$ but less than $(554)^2$. If in the given number, we add 4416 – 3952 = 464, then the sum will be a perfect square.

In word problems where number to be **multiplied** or **divided** has to be calculated so that the given number becomes a perfect square, prime factorization method of finding square root is used.

But, in word problems where number to be **added** or **subtracted** has to be calculated so that given number becomes a perfect square, long division method of finding square root is used.

7. Finding Square Roots of Decimal Numbers

We have

$(0.4)^2$	=	0.16, therefore	$\sqrt{0.16} = 0.4$
$(0.41)^2$	=	0.1681, therefore	$\sqrt{0.1681} = 0.41$
$(4.1)^2$	=	16.81, therefore	$\sqrt{16.81} = 4.1$

From these examples, observe that.

(i) Square of a decimal number consists of twice as many decimals places as given in the number.
(ii) The number of decimal places in the square root of a given decimal number is half of the number of decimal places in the given number.

Let us illustrate the method of finding square root of a decimal number (using long division method) through the following example. If the number of decimal places is odd, make them even by suffixing a zero.

Example: Find the square root of 37.0881

Solution:

Step 1: Place bars on the integral part as usual $\overline{37}.0881$

Step 2: Place bars on the decimal part on every pair of digits beginning with the first decimal place
$\overline{37}.\overline{08}\,\overline{81}$

Step 3: Start finding the square root as usual, (by long division method)

	6.___
6	$\overline{37}$. $\overline{08}$ $\overline{81}$
	36
12___	108

Step 4 : A decimal point is placed in the square root as soon as the integral part is exhausted.

	6 . 09
6	$\overline{37}$. $\overline{08}$ $\overline{81}$
	36
120	108
	000
1209	10881
	10881
	0

Step 5 : Complete the process as usual.

So , $\sqrt{37.0881} = 6.09$

Alternatively, we can find $\sqrt{37.0881}$ as

$$37.0881 = \frac{370881}{10000} = 0.0001 \times 370881$$

$$= (.01 \times .01) \times 370881$$

So, $\sqrt{37.0881} = \sqrt{(.01)(.01)} \times \sqrt{370881}$

$= (.01) \times 609 = 6.09$

Example: Find $\sqrt{.00053361}$

Solution:

	0.0231
2	0.$\overline{00}$ $\overline{05}$ $\overline{33}$ $\overline{61}$
	4
43	1 33
	1 29
461	4 61
	4 61
	0

So, $\sqrt{.00053361} = 0.0231$

8. ESTIMATING A SQUARE ROOT

Let the students estimate $\sqrt{125}$. For this, we have to think of two numbers which are **squares** and are **close to 125**.

Now, 121 < 125 < 144. where $\sqrt{121}$ = 11, $\sqrt{144}$ = 12

Or 11 < $\sqrt{125}$ < 12

So, $\sqrt{125}$ is closer to 11.

Thus, estimated value of $\sqrt{125}$ is 11.

Ask the students to estimate the following:

(i) $\sqrt{80}$ (ii) $\sqrt{1000}$ (iii) $\sqrt{200}$

Common Errors

(i) Some students may write

(a) $\sqrt{4} = \pm 2$.

But = 2 not – 2

(b) $\sqrt{-4}$ = – 2 or + 2 since (– 2) (– 2) = 4 and (+2) (+ 2) = 4

(c) = 0.2

(ii) $\sqrt{9} = \pm 3$

(iii) $\sqrt{0.9} = 0.3$

(iv) $\sqrt{16+9} = \sqrt{16} + \sqrt{9}$

(v) $\sqrt{100-64} = \sqrt{100} - \sqrt{64}$

(vi) $\sqrt{361} = \sqrt{36}\sqrt{1}$

(vii) $\sqrt{-16} = -4$

(viii) Number of digits in a square of 2 digit number is always four.

(ix) $\sqrt{x^2} = x$ for all values of x.

(x) 2, 3, 5 is a Pythagorean triplet since 2 + 3 = 5 .

(xi) 2.5, 6.0 and 6.5 is a Pythagorean triplet because $(6.5)^2 = (2.5)^2 + (6)^2$ is true.

(xii) $1^2 = (-1)^2$, therefore 1 = 1

Through the following exercise you may evaluate learning of the concepts.

Exercise

1. Are the following numbers perfect squares? Give reason.

   (i) 32 (ii) 100 (iii) 330550 (iv) 343

   (v) 7928 (vi) 1762

2. What will be the unit digits of the squares of the following numbers?

   (i) 22 (ii) 789 (iii) 3835 (iv) 146

3. Squares of which of the following would be odd numbers?

   (i) 421 (ii) 2816 (iii) 209

4. Squares of which of the following would be even numbers?

   (i) 420 (ii) 241 (iii) 1896

5. How many whole numbers lie between the squares of the following numbers?

   (i) 13 and 14 (ii) 50 and 51

6. Find the square of the following numbers .

   (i) 23 (ii) 81

7. Write a Pythagorean triplet whose one member is

(i) 4 (ii) 16 (iii) 12

8. Find the following square roots:

(i) (ii) $\sqrt{0.09}$

9. Find the square root of the following by prime factorization method:

(i) 900 (ii) 7056

10. Find the least number by which 180 must be multiplied so that the product is a perfect square.

11. Find the smallest number by which 9408 must be divided so that it becomes a perfect square.

12. Find the square roots of the following numbers by long division method :

(i) 2304

(ii) 27225

(iii) 390625

13. Find the square roots of the following numbers:

(i) 16.81 (ii) 9.3025

14. Find the square roots of the following numbers:

(i) 0.09 (ii) 0.0004

15. Write True or False for the following statements.

(i) The number of digits in a square number is even

(ii) The sum of two square numbers is a square number

(iii) The product of two square numbers is a square number

(iv) No square number is negative

(v) $\sqrt{9} = \pm 3$

(vi) $\sqrt{0.9} = 0.3$

Unit 7

Cubes and Cube Roots

Structure

- Introduction
- Main Concepts and Sub-concepts
- Objectives
- Teaching Points
  1. Cube numbers or perfect cubes
  2. Properties and patterns of cube numbers
  3. Cube roots
     - 3.1 Need for cube roots
     - 3.2 How to find cube root of a perfect cube?
       - 3.21 Cube root through prime factorisation method
       - 3.2.2 Cube root through a pattern
       - 3.2.3 Cube root using one's or unit's digit
- Common Errors
- Exercise

Introduction

Recall that if a natural number is a square of some natural number, it is called a **perfect square** or a **square number** and **square root** of a given natural number *n* is that number which when multiplied by itself gives *n* as a product. For example

$2^2 = 2 \times 2 = 4$, square root of 4 is 2

$3^2 = 3 \times 3 = 9$, square root of 9 is 3

$4^2 = 4 \times 4 = 16$, square root of 16 is 4, etc.

The numbers 4, 9, 16 are the square numbers and 2, 3, and 4 respectively are the square roots of 4, 9 and 16. We also know that

$2^3 = 2 \times 2 \times 2 = 8$

$3^3 = 3 \times 3 \times 3 = 27$

$4^3 = 4 \times 4 \times 4 = 64$, etc.

Numbers such as 8, 27, 64 are referred to as **perfect cubes** (or **cubes**) of the numbers 2, 3, 4, respectively.

In this unit, we will discuss about **cubes** and **cube roots** of numbers.

Main Concepts and Sub-concepts

- Cube numbers or perfect cubes and their properties
- Cube roots of perfect cubes through prime factorisation method
- Cube roots of cube numbers (having at most six digits)

Objectives

After teaching this unit, the student will be able to

- understand the meaning of a cube number or a perfect cube;
- find whether a given number is a cube number or not;
- find some interesting patterns in cube numbers;

- understand the meaning of a cube root of a number;
- find a cube root of number through prime factorisation method; and
- find cube root of a cube number (having at most six digits) by inspecting its digits.

Teaching Points

1. CUBE NUMBERS OR PERFECT CUBES

You may introduce the concept of a cube number by asking the students to recall the volume of a cube with side, say 1 cm.

Volume = 1cm   1cm × 1cm = 1^3 cm^3

Volume of a cube with side 2 cm = 2cm × 2cm × 2cm = 2^3 cm^3

Volume of a cube with side 3 cm = 3cm × 3cm × 3cm = 3^3 cm^3

and so on. Ask them to complete the following table:

Table 1

Side of a cube (in cm)	Volume (in cm^3)
1	$1 \times 1 \times 1 = 1^3$
2	$2 \times 2 \times 2 = 2^3 = 8$
3	$3 \quad 3 \times 3 = 3^3 = 27$
4	$4 \quad 4 \times 4 = 4^3 = 64$
5	$5 \times 5 \times 5 = 5^3 = 125$
6	____________________
7	____________________
8	____________________
9	
10	$10 \times 10 \times 10 = 10^3 = 1000$

Numbers like $1 = 1 \times 1 \times 1$, $8 = 2 \times 2 \times 2$, $27 = 3 \times 3 \times 3$, ———————— , $1000 = 10 \times 10 \times 10$, etc. are the numbers which can be expressed as the product of a number by itself three times .

Such numbers as 1, 8, 27, - - - , 1000 are known as **cube numbers or perfect cubes.**

> In general, a natural number is said to be a cube number or perfect cube if it is the cube of some natural number.
>
> In other words, a natural number n is said to be a cube number if $n = m \times m \times m = m^3$ for some natural number m.

How to know whether a given number is a cube number or not ?

Write the number as a product of its prime factors. If the number is a cube number, then you would be able to create triplets of the same factors.

If no factor is left out, the number is a cube number. However, if some factor is left out as a single or a double factor, then the number is **not** a cube number.

That is a given natural number is a perfect cube if it can be expressed as the product of three equal factors.

Let us illustrate it through examples.

Example: Examine whether 216 is a cube number or not

Solution: $216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3$ ⟵ prime factorisation

$= 2 \times 2 \times 2 \times \underbrace{3 \times 3 \times 3}$ ⟶ triplets of same factors

No single or double factor is left. Thus, 216 is a cube number.

Example: Is 128 a cube number?

Solution: Prime factorisation

$$128 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

Triplets of same factors

Remaining single factor

Among the prime factors of 128, one factor 2 does not appear in a group of 3. So, 128 is not a cube number.

2. PROPERTIES AND PATTERNS OF CUBE NUMBERS

(i) Ask the students to extend Table 1 for sides of a cube (in cm) upto 20.

(ii) Ask them to observe the table and answer the following questions:

(a) Is the cube of an odd number odd?

(b) Is the cube of an even number even?

Observe that $1^3 = 1$ (odd) $2^3 = 8$ (even)

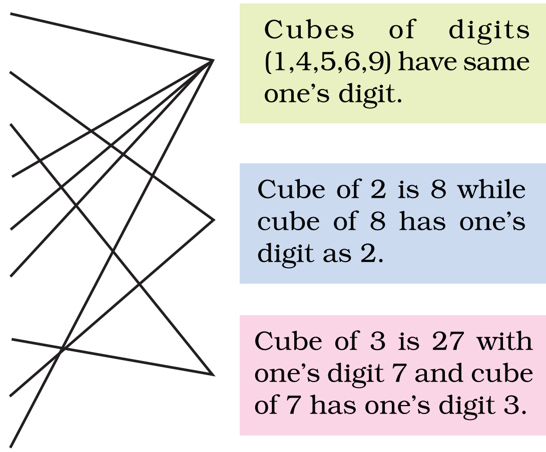
$3^3 = 27$ (odd) $4^3 = 64$ (even)

$5^3 = 125$ (odd) $6^3 = 216$ (even) and so on.

In general, **cubes of odd numbers are always odd** and **cubes of even numbers are always even**.

(iii) Ask the students to observe units digit or ones digit of cube numbers.

Cube number	**Units digit**
1^3 (= 1)	1
2^3 (= 8)	8
3^3 (= 27)	7
4^3 (= 64)	4
5^3 (= 125)	5
6^3 (= 216)	6
7^3 (= 343)	3
8^3 (= 512)	2
9^3 (= 729)	9



We find:

(i) Cubes of numbers with one's digit 1, 4, 5, 6 and 9, have same one's digit 1, 4, 5, 6 and 9, respectively.

(ii) Cube of a number with one's digit 2 has one's digit 8 while cube of a number with one's digit 8 has unit digit 2.

(iii) Cube of a number with one's digit 3 has one's digit 7 while cube of a number with one's digit 7 ends in 3.

(iv) Cube of a number with one's digit 0 has one's digit 0.

Number and its cube

Last digit of number	Last digit of the corresponding cube
0	0
1	1
2	8
3	7
4	4
5	5
6	6
7	3
8	2
9	9

Let the students observe:

$(-1) \times (-1) \times (-1) = (-1)^3 = -1$

$(-2) \times (-2) \times (-2) = (-2)^3 = -8$

$(-3) \times (-3) \times (-3) = (-3)^3 = -27$

$(-4) \times (-4) \times (-4) = (-4)^3 = -64$ and so on

1, –8, 27, –64 are cube numbers and are also negative, i.e. **negative numbers may also be cube numbers**.

What about square numbers? Can square numbers be negative? (Let the students think and come up with answers.)

(iv) Cube numbers are obtained on adding consecutive odd numbers

$1 = 1 = 1^3$

$3 + 5 = 8 = 2^3$

$7 + 9 + 11 = 27 = 3^3$

$13 + 15 + 17 + 19 = 64 = 4^3$

....

How many consecutive odd numbers will be needed to obtain the sum 5^3 ?, 6^3 ?, 7^3 ?

Let the students answer these questions.

Example: Is 675 a cube number? What is the smallest number by which it should be multiplied so that the product is a cube number?

Solution: $675 = 3 \quad 3 \times 3 \times 5 \times 5$ ← Prime factorisation

$= \underbrace{3 \times 3 \times 3}_{} \times \underbrace{5 \times 5}_{}$

↓ 3 occurs in a group of 3

↓ 5 does not occur in a group of 3

3	675
3	225
3	75
5	25
	5

So, 675 is **not** a cube number

Had there been one more factor 5 in the prime factorisation, then the number would have been a cube number.

So, if we multiply 675 by 5, the product would be

$$\underbrace{3 \times 3 \times 3}_{\text{3 occurs in a group of 3}} \times \underbrace{5 \times 5 \times 5}_{\text{5 occurs in a group of 3}}$$

and we get a cube number. Note that 5 is the **smallest number** to be multiplied in this case.

Example: What is the smallest natural number by which 675 may be divided so that the **quotient** is a cube number?

Solution: $675 = \underbrace{3 \times 3 \times 3}_{\text{3 occurs in a group of 3}} \times \underbrace{5 \times 5}_{\text{5 does not occur in a group of 3}}$ ← Prime factorisation

If we divide by 5×5 (= 25), the quotient would be $3 \times 3 \times 3$, which is a cube number. So, we must divide 675 by 25 so that the quotient is a cube number. Can there by any other number less than 25? Let the student think and you may discuss with the students.

(i) Ask the students to do the following:

(a) How many cube numbers or perfect cubes, in all are there from1 to 1000?

(b) Write the one's digit in the cube number of each of the following:

(i) 31 (ii) 109 (iii) 4447

(iv) 592 (v) 125125

(c) Is 243 a perfect cube?

(d) Is 392 a perfect cube? If not, what is the smallest number by which this should be multiplied (or divided) so that the product (or quotient) is a perfect cube?

3. CUBE ROOTS

3.1 NEED FOR CUBE ROOTS

If the area of a floor (square in shape) is 121 m^2, what would be the length of its side? To answer this question, we have to find a number whose square is 121. In the same way, if the volume of a cube is 216 cm^3, what would be the length of the side of the cube?

To answer this question, we have to find a number whose cube is 216. Such number is called **cube root** of 216. Now we will discuss how to find **cube root** of a **cube number**.

Let the students observe Table 1.

$1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, ...

$1^3 = 1$, we say **cube root** of 1 is **1**

$2^3 = 8$, we say **cube root** of 8 is **2**

$3^3 = 27$, we say **cube root** of 27 is **3**

$4^3 = 64$, we say **cube root** of 64 is **4** and so on.

Cube root of a number is denoted by the symbol $\sqrt[3]{\;}$

Thus, cube root of 8 is denoted as $\sqrt[3]{8}$

Cube root of 27 is denoted as $\sqrt[3]{27}$ and so on. Cube root of a cube number '*a*' is denoted as $\sqrt[3]{a}$

A natural number m is the cube root of a number *n*, if $n = m^3$

3.2 HOW TO FIND CUBE ROOT OF A PERFECT CUBE?

3.2.1 Cube root through prime factorisation method

Let us find cube root of 343.

As done in case of finding a square root of a perfect square, we first express 343 in the form of product of prime factors.

$343 = 7 \times 7 \times 7$ ⟵ Prime factorisation

$= 7 \quad 7 \times 7$ ← 7 occurs in a group of 3 and no factor occurs in a group of two or one.

So, **cube root** of 343 is 7

or $= 7$

Example: Find cube root of 1728

Solution: Prime factorisation

$$1728 = 2\times2\times2 \times 2\times2\times2 \times 3\times3\times3$$

$$= \underbrace{2\times2\times2} \times \underbrace{2\times2\times2} \times \underbrace{3\times3\times3}$$

Each one is in a group of three. No factor is left which is in a group of two or one

That is 1728 $= 2^3 \times 2^3 \times 3^3 = (2 \times 2 \times 3)^3$

Thus, cube root of 1728 $= 2 \times 2 \times 3$

$= 12$

or $= 12$

Check: $12 \times 12 \times 12 = 1728$.

3.2.2 Cube root through a pattern (using successive subtraction)

We have	Also
$1^3 = 1$	$1 = 1^3$
$2^3 - 1^3 = 7$	$1 + 7 = 2^3$
$3^3 - 2^3 = 19$	$1 + 7 + 19 = 3^3$
$4^3 - 3^3 = 37$	$1 + 7 + 19 + 37 = 4^3$
$5^3 - 4^3 = 61$	$1 + 7 + 19 + 37 + 61 = 5^3$
$6^3 - 5^3 = 91$	$1 + 7 + 19 + 37 + 61 + 91 = 6^3$
and so on	and so on.

Thus, to find the cube root of a perfect cube number, we have to subtract successively 1, 7, 19, 37, ... from the given number till we get a zero. The number of times the subtraction is carried out gives the cube root. Let us see this through an example.

Example: Find the cube root of 216

Solution:

216 – 1 = 215 → 1st number

215 – 7 = 208 → 2nd number

208 – 19 = 189 → 3rd number

189 – 37 = 152 → 4th number

152 – 61 = 91 → 5th number

91 – 91 = 0 → 6th number

$\overline{216}$

The numbers 1, 7, 19, 37, 61, 91, ... can be obtained by putting n = 1, 2, 3, 4, 5, 6,......... in $1 + n(n - 1) \times 3$

Since we have subtracted **six** times, = 6

Example: Examine if 400 is a perfect cube .

Solution:

400 – 1 = 399

399 – 7 = 392

392 – 19 = 373

373 – 37 = 336

336 – 61 = 275

275 – 91 = 184

184 – 127 = 57

Put n =7 in $1+n(n - 1) \times 3$
$1 + 7(7 - 1) \times 3 = 127$

What will be the next number to be subtracted?

Put $n = 8$ in

$1 + n\ (n - 1)\quad 3$ to get

$1 + 8\ (8 - 1) \times 3 = 169$

If we subtract 169 from 57, it does not give zero. So, 400 is **not a perfect cube**.

3.2.3 Cube root using one's or units digit

We know that cube of a number ending in 0, 1, 4, 5, 6, or 9 ends in 0,1, 4, 5, 6 or 9, respectively. The cube of number ending in 2 ends in 8 and vice versa. Similarly, the cubes of numbers ending in 3 and 7 end in 7 and 3, respectively. Consider a number say 512.

Let us find the cube root by using the facts given above

(a) *Examine the units digit*

Number : 512

→ Unit's digit is 2

↓

So, the digit at the unit's place in the cube root of 512 is **8**

(b) Strike out from the right, last three (i.e. units, tens and hundreds) digits of the number. If nothing is left, the digit in Step (a) is the cube root.

Since no number is left after striking out the units, tens and hundreds digits, so **8** is the cube root.

Check : $8 \times 8 \times 8 = 512$

Example: Find the cube root of 2197.

Solution: 2 197

→ One group of three digits.
→ Unit's digit is 7

↓

So, the digit at the unit's place in the cube root of 2197 is 3. (Cube of a number ending in 3 ends in 7)

2~~197~~ (strike out units, tens and hundreds digits)

→ 2 is left

Now 1 is the largest number whose cube is less than 2.
So, the ten's digit of the cube root = 1
Thus, the required cube root = **13**
Check : 13 × 13 × 13 = 2197

Example: Find the cube root of 117649.

Solution: 117 649

→ Unit's digit is 9

↓

So, the digit at the unit's place in the cube root of the given number is 9 [Think. Why?]

117 ~~649~~

↓

(Striking out last three digits)

Now 4^3 (= 64) < 117 < 5^3 (=125>117)

Hence, the tens digit of the cube root is 4.

Thus, the required cube root is **49**

Check : 49 × 49 × 49 = 117649

Let the students find the cube roots of the following numbers:

(i) 729 (ii) 9261	by prime factorisation method of section 3.2.1
(iii) 125 (iv) 729	through the pattern using successive subtraction as discussed in section 3.2.2 above
(v) 91125 (vi) 157464	using the method given in section 3.2.3

Example: Meenakshi has some wooden cuboids of sides 7cm, 3cm, 7cm. Atleast how many such cuboids will she need to make a cube?

Solution: Number of cuboids × Volume of one cuboid = Volume of cube formed

Number of cuboids × (7 × 3 × 7) = a perfect cube .... (1)

(We know that volume of cube = $(\text{side})^3$, so it has to be a perfect cube)

Now, since the RHS is a perfect cube, LHS should also be a perfect cube. Now, to proceed further, teacher should raise a question, by what number 7 × 3 × 7 be multiplied to make it a perfect cube.

Obviously, to make 7 × 3 × 7 a perfect cube, it should be multiplied by 3 × 3 × 7 so (1) can be written as,

$$(3 \times 3 \times 7) \times (7 \times 3 \times 7) = \text{a perfect cube}$$

$$\therefore \text{ Number of cuboids } = 3 \times 3 \times 7$$

$$= 63$$

Note: Teacher can help the students to find the length of side of cube so obtained.

Volume of cube = $3^3 \times 7^3 = (3 \times 7)^3 = (21)^3$

$\therefore$ Side of cube = 21cm.

Common Errors

1. Students sometimes use the symbol $3\sqrt{\ }$ for cube root of the number in place of $\sqrt[3]{\ }$, which is not correct. The first one stands for 3 times the square root of the number not for the cube root of the number
2. Students may write cube root of 27 as $\pm$ 3.

   (–3) is not the cube root of 27 as $(-3)^3 = -27$ not 27.
3. Square of a number is positive, so cube of that number will also be positive.
4. There is no cube root of a negative number.
5. $\sqrt[3]{8} = \pm 2$
6. $\sqrt[3]{8+27} = \sqrt[3]{8} + \sqrt[3]{27}$

Through the following exercise, you may evaluate learning of the concepts.

Exercise

1. Write the units digit of the cubes of the following numbers:

   (i) 249 (ii) 152 (iii) 2177
2. Which of the following numbers are not perfect cubes ?

   (i) 2744 (ii) 46656
3. What is the smallest number by which 8640 be multiplied so that the product is a perfect cube?
4. What is the smallest number by which 3087 be divided so that the quotient is a perfect cube?
5. Find the cube root of the following numbers by prime factorization method.

   (i) 3375

   (ii) 8000

6. Find the cube root of 2197 through the pattern using successive subtraction discussed in 3.2.2.

7. Find the cube root of

   (i) 91125 (ii) 166375

   by using the method discussed in 3.2.3, i.e. by using unit's digit method.

8. Write True or False for the following statements.

   (i) Cube of an odd number is odd.

   (ii) Cube of an even number is odd.

   (iii) Cube of an odd number is even.

   (iv) Cube of an even number is even.

   (v) If a number ends in 7, its cube will also end in 7.

   (vi) There is no perfect cube which ends with exactly two zeros.

   (vii) If a number ends in 2, its cube will end in 8.

   (viii) If a number ends in 9, its cube root will end in 3.

Unit 8

Powers and Exponents

Structure

- Introduction
- Main Concepts and Sub-concepts
- Objectives
- Teaching Points

  1. Powers (Base as natural number)

     1.1 Base as a negative integer

     1.2 Base as an integer

  2. Laws of Exponents

     2.1 Product of powers with the same base

     2.2 Quotient of powers with the same base

     2.3 Power of a power

     2.4 Product of powers with the same exponents

     2.5 Quotient of powers with the same exponents

     2.6 Numbers with exponent zero

3. Expressing Large Numbers in the Standard Form
4. Powers with Negative Exponents
5. Laws of Exponents When Exponent(s) is/are Negative
6. Expressing Small Numbers in the Standard Form
7. Points to be Highlighted

- Common Errors
- Exercise

Introduction

The teacher may take the help of the following story to motivate the students regarding the need of exponents.

In the game of chess, the chessboard has 64 squares. The King of Persia was so pleased with the inventor of the game that he offered him any reward that the inventor wanted. The inventor asked to give him as much wheat as is required in the following way: One grain of wheat be placed on the first square, two times of the number of grains on the second square, i.e. 2 on the 2nd square, two times the number of grains on the second square, i.e. 4 on the third square and so on. You may now start asking questions:

How many grains of wheat on 3rd square?

How many on 4th square? How many on 10th square? And so on.

Finally, how many grains of wheat on 64th square?

Students may answer like this:

Number of grains on 3rd square

= 2 times the number of grains on 2nd square

$= \quad 2 \times 2$

Similarly, the number of grains on 4th square

$= \quad 2 \times 2 \times 2$

And the number of grains of wheat on 10th square

$= \quad 2 \times 2 \quad 2 \times 2 \quad 2 \times 2 \times 2 \times 2 \quad 2$

and the number of grains of wheat on 64th square

$= \quad 2 \quad 2 \times 2 \times \ldots\ldots \times 2$

63 times

Tell the students that there is a need to find ways to express these products in a shorter way by introducing the concept of what we call **exponents** and **powers**. In this unit, we will discuss exponents and powers of the numbers.

Main Concepts and Sub-concepts

- Exponents, power of a number, exponential form of a number.
- Laws of exponents: For non-zero integers *a* and *b* and whole numbers *m* and *n*,
  - (1) $a^m \quad a^n = a^{m+n}$ (product of powers with same base)
  - (2) $a^m \div a^n = a^{m-n}$ (where $m > n$) (quotient of powers with same base)
  - (3) $(a^m)^n = a^{mn}$ (power of a power)
  - (4) $a^m \times b^m = (ab)^m$ (product of power with same exponent)
  - (5) $a^m \div b^m = (a \div b)^m$ (quotient of powers with same exponent)
- Large numbers expressed in standard form.
  - (1) Powers with exponents as integers
  - (2) Laws of exponents stated above for non-zero integers *a* and *b* and integers *m* and *n*
  - (3) Very small and very large numbers expressed in standard form
  - (4) Comparison of very small and very large numbers

Objectives

After teaching these concepts/sub-concepts, the students can

- understand the meaning of an exponent, power of number, exponential form of a number;
- distinguish between power and exponent;
- understand the laws of exponents and use them in problems involving powers/exponents;
- use exponents in expressing a
  - (i) very large number in standard form and vice versa
  - (ii) very small number in standard form and vice versa; and
- compare very large numbers and very small numbers using powers and exponents.

Teaching Points

1. POWERS

Let the students recall that a repeated addition is called multiplication

$$2 + 2 = 2 \times 2$$
$$2 + 2 + 2 = 2 \times 3$$
$$2 + 2 + 2 + 2 = 2 \times 4$$
$$2 + 2 + 2 + 2 + 2 = 2 \times 5$$

Number of times '2' occurs in repeated addition, and so on.

Ask the students to try to write the following repeated multiplications in a short way:

$2 \times 2, \quad 2 \times 2 \times 2, \quad 2 \times 2 \times 2 \times 2,$

$2 \times 2 \times 2 \times 2 \times 2, \ldots$

Help them to write these repeated multiplications as

$$2 \times 2 = 2^2$$
$$2 \times 2 \times 2 = 2^3$$
$$2 \times 2 \times 2 \times 2 = 2^4$$
$$2 \times 2 \times 2 \times 2 \times 2 = 2^5$$

They indicate the number of times '2' has been multiplied with itself

The short notation for writing repeated multiplication is known as **exponential form**.

Here 2^2, 2^3, 2^4, 2^5 are all in exponential forms.

2^3 is the product of 2 three times. It is referred to as **the third power** of 2. 3 is called the **exponent** and 2 is called the base. 2^3 is read as **2 raised to the power 3** or 2 cubed.

Similarly, in 2^5, exponent is 5 and base is 2

in 2^4, exponent is 4 and base is 2

In the same way,

(i) 3^4 is the product of four 3s. Here exponent is 4 and base is 3.

(ii) 10^7 is the product of seven 10s. Here exponent is 7 and the base is 10. It is called the seventh power of 10 or 10 raised to the power 7.

Now you may go back to the story of chess given in 'introduction'. By using exponents, we can now write:

The number of grains at the 2nd square = 2

The number of grains at the 3rd square = $2 \times 2 = 2^2$

The number of grains at the 4th square = $2 \times 2 \times 2 = 2^3$

The number of grains at the 10th square = $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9$ and so on.

The number of grains at the 64th square =

$$\underbrace{2 \times 2 \times 2 \ldots \times 2}_{63 \text{ times}} = 2^{63}$$

Let the student feel how conveniently we have expressed the number of grains at the 64th square of the chess .

In its applications, the students often see exponents when finding area or volume. For example, the area of a square floor with side 7 metres is

$$7\text{m} \times 7\text{m} = 7^2\ \text{m}^2 = 49\ \text{m}^2$$

The volume of a cube whose edge measures 7 cm is 7cm × 7cm × 7cm $= 7^3 \text{ cm}^3$
$= 343 \text{ cm}^3$

Special names are given to numbers with exponents 2 and 3. 7^2 is referred to as 7 **squared** and 7^3 is referred to as 7 **cubed**. There are no special names for other powers.

1.1 BASE AS A NEGATIVE INTEGER

We can extend the concept to numbers with base to be a negative integer.
In this context, what would $(-3)^2$ mean? It is $(-3) \times (-3)$.
What does $(-5)^5$ mean ?
It is $(-5) \times (-5) \times (-5) \times (-5) \times (-5)$.

1.2 BASE AS AN INTEGER *a*

Instead of taking base as a fixed number, let it be any integer a. Then
$a \times a = a^2$ (read as 'a squared' or 'a raised to the power 2')
$a \times a \times a = a^3$ (read as 'a cubed' or 'a raised to the power 3')

$a \times a \times a \times a = a^4$ (read as '4^{th} power of a' or a raised to the power 4) and so on.

Similarly, $a \times a \times a \times b \times b$ can be expressed as a^3b^2 (read as a cubed b squared)

$a \times a \times b \times b \times b \times b$ can be express as $a^2 b^4$ (read as a squared into b raised to the power 4)

Example: Express 64 as a power of 2.

Solution: $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$

So, $64 = 2^6$

Example: Find (i) $(5)^5$ (ii) $(-5)^5$

Solution: (i) $(5)^5 = 5 \times 5 \times 5 \times 5 \times 5 = 3125$

(ii) $(-5)^5 = (-5) \times (-5) \times (-5) \times (-5) \times (-5) = -3125$

Example: Write $a \times a \times a \times a \times b \times b \times b$ in exponential form

Solution: $\underbrace{a \times a \times a \times a}_{4 \text{ times}} \times \underbrace{b \times b \times b}_{3 \text{ times}} = a^4 \times b^3$

2. LAWS OF EXPONENTS

2.1 PRODUCT OF POWERS WITH THE SAME BASE

Main Result: For any non-zero integer a and positive integers m and n

$$a^m \times a^n = a^{m+n} \qquad \text{.....................(1)}$$

Before arriving at this law, teacher should gradually introduce it through examples. For example, ask the students to multiply 2^3 and 2^5.

$$2^3 \times 2^5 = \underbrace{(2 \times 2 \times 2)}_{3 \text{ times}} \times \underbrace{(2 \times 2 \times 2 \times 2 \times 2)}_{5 \text{ times}}$$

$$= \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{8 \text{ times } = (3 + 5) \text{ times}}$$

$$= 2^8$$

Give one more such example and one example in which base is a negative integer. Then proceed to (1) in the following manner:

$$a^m = (a \times a \times a \ldots \times a)\ (m \text{ times})$$

$$a^n = (a \times a \times a \ldots \times a)\ (n \text{ times})$$

$$a^m \times a^n = \underbrace{(a \times a \times a \ldots \times a)}_{(m + n) \text{ times}} = a^{m+n}$$

2.2 QUOTIENT OF POWERS WITH THE SAME BASE

Main Result: For any non-zero integer a, and positive integers m and n with $m>n$,

$$a^m \div a^n = a^{m-n} \qquad \text{.......................... (2)}$$

Again before arriving at this law, teachers should gradually introduce it through examples. Ask the students to find $3^6 \div 3^2$

$$3^6 = 3 \times 3 \times 3 \times 3 \times 3 \times 3$$

$$3^2 = 3 \times 3$$

$$3^6 \div 3^2 = \frac{3\times3\times3\times3\times3\times3}{3\times3} = 3 \times 3 \times 3 \times 3$$

$$= 3^4 = 3^{6-2}$$

Now, $\frac{3^2}{3^2} = \frac{3\times3}{3\times3} = 1$

Also, $\frac{3^2}{3^2} = 3^{2-2} = 3^0 \qquad \left\{ \text{As} \frac{a^m}{a^n} = a^{m-n} \right\}$

So, $3^0 = 1$

Thus $1 = \frac{a^m}{a^m} = a^{m-m} = a^0$ Therefore, $a^0 = 1$

Similarly

$$(-4)^5 \div (-4)^3 = \frac{(-4)\times(-4)\times(-4)\times(-4)\times(-4)}{(-4)\times(-4)\times(-4)}$$

$$= (-4) \times (-4)$$

$$= (-4)^2 = (-4)^{5-3}$$

The teacher may highlight that $m > n$ here.

2.3 POWER OF A POWER

Main Result: For any non-zero integer a and positive integers m and n :

$$(a^m)^n = a^{mn} \qquad \text{..........................} (3)$$

Again introduce this law, i.e. (3) through examples. First explain the meaning of the expressions, $(a^m)^n$ and a^{mn} through examples such as $(2^3)^4$, $(3^2)^5$, etc.

$(2^3)^4$ means (2^3) is multiplied four times with itself, i.e.

$$
\begin{aligned}
(2^3)^4 &= (2^3) \times (2^3) \times (2^3) \times (2^3) \\
&= (2^{3+3}) \times (2^{3+3}) && \text{[Using Law (1)]} \\
&= 2^6 \times 2^6 = 2^{6+6} && \text{[Using Law (1)]} \\
&= 2^{12} = 2^{3 \times 4}
\end{aligned}
$$

$$
\begin{aligned}
\text{Similarly } (3^2)^5 &= (3^2) \times (3^2) \times (3^2) \times (3^2) \times (3^2) \\
&= (3^{2+2}) \times (3^{2+2}) \times 3^2 \\
&= 3^4 \times 3^4 \times 3^2 \\
&= 3^{4+4} \times 3^2 \\
&= 3^8 \times 3^2 \\
&= 3^{10} = 3^{2 \times 5}
\end{aligned}
$$

$$\text{Similarly } (7^2)^9 = 7^{2\times 9} = 7^{18}$$

2.4 PRODUCT OF POWERS WITH THE SAME EXPONENTS

Main Result : For non-zero integers a, b and positive integer m :

$$\boxed{a^m \times b^m = (ab)^m} \qquad (4)$$

Introduce this law through examples such as

$$
\begin{aligned}
2^3 \times 3^3 &= (2 \times 2 \times 2) \times (3 \times 3 \times 3) \\
&\quad \text{(By definition)} \\
&= (2 \times 3) \times (2 \times 3) \times (2 \times 3) \\
&\quad \text{(by rearranging and grouping)} \\
&= (2 \times 3)^3
\end{aligned}
$$

$$
\begin{aligned}
4^2 \times 7^2 &= (4 \times 4) \times (7 \times 7) \\
&\quad \text{(By definition)} \\
&= (4 \times 7) \times (4 \times 7) \\
&\quad \text{(by rearranging and grouping)} \\
&= (4 \times 7)^2
\end{aligned}
$$

$$\text{Similarly, } 3^2 \times 2^2 = (3 \times 2)^2 = 6^2$$

2.5 QUOTIENT OF POWERS WITH THE SAME EXPONENTS

Main Result : For non-zero integers *a*, *b* and positive integer *m* :

$$a^m \div b^m = \qquad (5)$$

Introduce Law (5) through examples such as

$$3^4 \div 2^4 =$$

$$= \left(\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\right)$$

$$= \left(\frac{3}{2}\right)^4$$

$$5^3 \div 8^3 =$$

$$= \frac{5}{8}\times\frac{5}{8}\times\frac{5}{8}$$

$$= \left(\frac{5}{8}\right)^3$$

Similarly, $3^9 \div 7^9 = \left(\frac{3}{7}\right)^9$

2.6 NUMBERS WITH EXPONENT ZERO

Let the students observe the following pattern:

2^6	=	2×2×2×2×2×2	=	64		
2^5	=	2×2×2×2×2	=	32	←	(64 ÷ 2)
2^4	=	2×2×2×2	=	16	←	(32 ÷ 2)
2^3	=	2×2×2	=	8	←	(16 ÷ 2)
2^2	=	2×2	=	4	←	(8 ÷ 2)
2^1	=	2	=	2	←	(4 ÷ 2)
2^0	=	?	=	?	←	(2 ÷ 2)

Through this pattern $\boxed{2^0 = 1}$

Some more patterns:

3^3	=	$3\times3\times3$	=	27	
3^2	=	3×3	=	9	$(27 \div 3)$
3^1	=	3	=	3	$(9 \div 3)$
3^0	=	?	=	?	$(3 \div 3)$

Here again, through pattern $\boxed{3^0 = 1}$

In fact, for any non-zero integer *a*, we define

$$\boxed{a^0 = 1} \qquad (6)$$

Now with the result (6), i.e. $a^0 = 1$ for any non-zero integer, we can now extend the Laws of Exponents (1), (2), (3), (4) and (5) for any whole numbers m and n.

Point to be emphasised

Laws (1) to (5) were introduced for exponents as positive integers m and n and base as non-zero integers a and b, but with $a^0 = 1$, for non-zero integer a, the laws are still valid for exponents as whole numbers and Law (2) is valid for $m \geq n$.

The teacher may give all types of questions involving base as **non-zero integer** and **exponents as whole numbers**.

Example: Simplify

(i) $7^2 \times 7^5$

(ii) $3^5 \div 3^2$

(iii) $(2)^5 \times (2)^6$

(iv) $(-4)^4 \div (-4)^3$

Solution:

(i) $7^2 \times 7^5 = 7^{2+5} = 7^7$ [Using Law (1)]

(ii) $3^5 \div 3^2 = 3^{5-2} = 3^3$ [Using Law (2)]

(iii) $(2)^5 \times (2)^6 = (2)^{5+6} = (2)^{11}$ [Using Law (1)]

(iv) $(-4)^4 \div (-4)^3 = (-4)^{4-3} = (-4)^1 = -4$ [Using Law (2)]

Example: Simplify

(i) $(3^2)^3 \times 2^6$

(ii) $\dfrac{3\times2^4\times3^5}{4\times9^2}$

Solution: (i) $(3^2)^3 \times 2^6 = 3^{2x3} \times 2^6$ [Using Law (3)]

$= 3^6 \times 2^6$

$= (3 \times 2)^6$ [Using Law (4)]

$= 6^6$

(ii) $\dfrac{3 \times 2^4 \times 3^5}{4 \times 9^2} = \dfrac{3 \times 2^4 \times 3^5}{4 \times (3^2)^2}$

$= \dfrac{3 \times 2^4 \times 3^5}{4 \times 3^4}$ [Using Law (3)]

$= \dfrac{3^{1+5} \times 2^4}{2^2 \times 3^4}$ [Using Law (1)]

$= \dfrac{3^6 \times 2^4}{2^2 \times 3^4}$

$= 3^{6-4} \times 2^{4-2}$ [Using Law (2)]

$= 3^2 \times 2^2$

$= (3 \times 2)^2$ [Using Law (4)]

$= 6^2 = 36$

Let the students do some more questions as given in Exercises 13.1 and 13.2 of Class VII Mathematics, NCERT.

When students are doing these exercises, you may insist that they should clearly mention laws of exponents applied in the steps of simplification. But later on, it may not be necessary.

3. EXPRESSING LARGE NUMBERS IN THE STANDARD FORM

In many situations, we come across numbers which are very large. For example:

(i) Speed of light in vacuum is 300,000,000 m/s.

(ii) Number of stars in our Galaxy is 100,000,000,000

(iii) Mass of Earth is 5,970,000,000,000,000,000,000,000 kg

(iv) The distance between Earth and Moon is 384,000,000 m

(v) Mass of Uranus is 86,800,000,000,000,000,000,000,000 kg, etc.

Such large numbers are **normally approximations**.

Ask the students to read these figures and let them compare the masses of Earth and Uranus. Which has greater mass, Earth or Uranus? Let the students feel difficulty in reading, understanding and comparing such large numbers. To overcome these difficulties, let us see how we can make use of the concepts of powers and exponents.

Such large numbers are usually written in shorter form using exponents with base 10.

For example, the number 300,000,000 may be written as $3 \times 100{,}000{,}000 = 3 \times 10^8$ or 30×10^7 or 300×10^6, etc. Thus, a very large number can be expressed as $k \times 10^n$, where k and n are some positive integers.

However, for the sake of uniformity, we write the number in the form $k \times 10^n$, taking k as a terminating decimal number such that $1 \leq k < 10$ and adjust n accordingly by expressing the numbers in this form, is referred to as expressed in **standard form**. Thus, the number 300,000,000 can be written in standard form as: 3×10^8.

Now , writing in standard form:

Mass of Earth = 5970,000,000,000,000,000,000,000 kg

$$= \underbrace{5.970}_{\substack{k \\ 1 \leq k < 10}} \times 10^{24} \text{ kg} \quad \text{.............(1)}$$

Mass of Uranus = 86,800,000,000,000,000,000,000,000 kg

$$= \underbrace{8.68}_{k,\ [1 \le k < 10]} \times 10^{25} \text{ kg} \quad \ldots\ldots\ldots\ldots(2)$$

Ask the students to read (1) and (2).

They can now compare both the numbers easily, i.e. they can compare the masses of Earth and Uranus. Let the students write the distance (in m) and number of stars given in (iv) and (ii) respectively above in standard form.

Example: Write the number 3186500000 in the standard form.

Solution: To write the number in the standard form, i.e. $k \times 10^n$, it is better if a student initially proceeds in the following way:

$$3186500000 = 31865 \times 10^5$$
$$= 3186.5 \times 10^6 = 318.65 \times 10^7$$
$$= 31.865 \times 10^8 = \underline{3.1865 \times 10^9}$$

$\underbrace{3.1865}_{k} \times 10^{\overset{\uparrow}{n}9}$ is the required standard form of the given number.

> Initially let the students proceed in the above way but later on they should write the numbers in standard form directly once they have understood the meaning of k and n.

Let the students do some more exercise as given in Exercise 13.3 of Class VII Mathematics, NCERT, 2007.

4. Powers with Negative Exponents

Introduce powers with negative exponents as given below:

3^2	=	3×3	$= 9$
3^1	=	3	$= 9 \div 3$
3^0	=	1 ←	$= 3 \div 3$
3^{-1}	=	? ←	$= 1 \div 3$

Let the students **observe the above pattern, as the exponent decreases by 1, the value becomes one-third of the previous value.**

So, $$3^{-1} = \frac{1}{3} = \frac{1}{3^1}$$

Similarly, $$3^{-2} = \frac{1}{3} \div 3 = \quad \times \quad = \frac{1}{9} = \frac{1}{3^2}$$

$$3^{-3} = \frac{1}{9} \div 3 = \quad \times \quad = \frac{1}{27} = \frac{1}{3^3}$$

and so on. Ask the students to find 10^{-2}, 2^{-5} in the same way by the following law:

> In general, for any non-zero integer *a*,
>
> $$a^{-m} = \frac{1}{a^m} \qquad (7)$$
>
> where *m* is a positive integer.

Also, a^{-m} is the multiplicative inverse of a^m. Students can now be encouraged to solve 'Try these' given on page 194 of Class VIII Mathematics, NCERT.

In view of Law (7), Law (2) can be extended to the case when $m < n$ also. Teacher has to pinpoint this.

5. LAWS OF EXPONENTS

When Exponent(s) is (are) Negative

Laws of exponents 2.1 to 2.6 as discussed in Section (2) of this unit are also valid when exponent is a negative integer, with Law (7).

In general, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$, if n is a positive integer.

Example: Find the value of

(i) 10^{-3} (ii) $\frac{1}{2^{-4}}$

Solution: (i) $10^{-3} = \frac{1}{10^3}$ [from Law (7)]

$= \frac{1}{1000}$

(ii) $\frac{1}{2^{-4}} = 2^4 = 16$ [from Law (7)]

Example: Simplify

(i) $(-3)^5 \div (-3)^6$

(ii) $\left(\frac{1}{5^2}\right)^3$

Solution: (i) $(-\ 3)^5 \quad (\quad 3\ )^6 = \quad = (-3)^{5-6}$

[Using Law (2)]

$= (-\ 3)^{-1} = \quad = \frac{-1}{3}$

[Using Law (7)]

(ii) $\left(\frac{1}{5^2}\right)^3 = (\ 5^{-2}\ )^3$ [Using Law (7)]

$= 5^{-2\text{x}3}$ [Using Law (3)]

$= 5^{-6} = \frac{1}{5^6}$ [Using Law (7)]

Example: Simplify $\left[\left\{\left(\frac{1}{4}\right)^{-2}-\left(\frac{1}{3}\right)^{-3}\right\}\div\left(\frac{1}{4}\right)^{-3}\right]$

Solution: To solve such a problem, proceed in the following way:

Step 1: $\left(\frac{1}{4}\right)^{-2}=\left(\frac{1^{-2}}{4^{-2}}\right)=\frac{1}{4^{-2}}=4^{2}$

Step 2: $\left(\frac{1}{3}\right)^{-3}=\left(\frac{1^{-3}}{3^{-3}}\right)=\frac{1}{3^{-3}}=3^{3}$

Similarly $\left(\frac{1}{4}\right)^{-3} = 4^{3}$

So, $\left[\left\{\left(\frac{1}{4}\right)^{-2}-\left(\frac{1}{3}\right)^{-3}\right\}\div\left(\frac{1}{4}\right)^{-3}\right]$

$=\left[\left\{4^{2}-3^{3}\right\}\div 4^{3}\right]$

$=\frac{4^{2}-3^{3}}{4^{3}}$

$=\frac{16-27}{64}=-\frac{11}{64}$

Let the student state the respective law applied in each step. But later on with sufficient practice, they may leave writing the laws. Some students will use one law and some may use another law in simplifying such expressions. But final answer will be the same. For example, in the above example, one student may also proceed as follows:

$$\frac{\left(\frac{1}{4}\right)^{-2}}{\left(\frac{1}{4}\right)^{-3}}-\frac{\left(\frac{1}{3}\right)^{-3}}{\left(\frac{1}{4}\right)^{-3}}=\left(\frac{1}{4}\right)^{-2+3} -$$

$$= \left(\frac{1}{4}\right)^{1} - \left(\frac{4}{3}\right)^{-3}$$

$$= \frac{1}{4} - \left(\frac{3}{4}\right)^{3}$$

$$= \frac{1}{4} - \frac{27}{64} = -\frac{11}{64}$$

Note that here, the student has used the laws when base is a non-zero integer. Of course these laws are also valid when base is a non-zero rational number. But at this stage, we have taken base as non-zero integer only. Tell the students that $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^{-m}$.

Let the students do more exercises as given in Exercise 12.1, of Class VIII Mathematics, NCERT.

6. EXPRESSING SMALL NUMBERS IN THE STANDARD FORM

In Section 3, we discussed how to write a very large number in the standard form of the type $k \times 10^n$, where k is a number such that $1 \leq k < 10$ and n is a positive integer.

Since a student has now learnt negative exponents, so, applying this knowledge she can also express very small numbers in the form

$k \times 10^{-n}$ where $1 \leq k < 10$, and n is a positive integer.

Let us illustrate it through an example.

Example: Write the number 0.0000837 in the standard form.

Solution: 0.0000837 =

$$= \frac{837}{10^7}$$

Student already knows how to write in this form

$= 837 \times 10^{-7}$

$= 83.7 \quad 10^{-6}$

$= 8.37 \times 10^{-5}$ **Standard Form**

Here $k = 8.37$ and $n = 5$

Initially, let the students proceed to write very small numbers in the standard form in the way given in this example. But later on, with sufficient practice, they should find k and n directly.

Example: Express the number appearing in the following statement in the standard form as:
charge of an electron is
0.000,000,000,000,000,00016 coulomb

Solution : $0.000,000,000,000,000,000,16 = 1.6 \times 10^{-19}$

Number of zeroes = 18

Let the students do some more exercise as given in Exercise 12.2 of Class VIII Mathematics, NCERT.

7. POINTS TO BE HIGHLIGHTED

(i) If a negative number has an odd power, then value of that number is negative. If a negative number has an even power, then the value of that number is positive.

(ii) Laws of exponents do not apply to multiplying or dividing of different powers if the bases are not same.

Example : $2^4 \times 3^5 = 16 \times 243 = 3888$

$2^4 \times 3^5$ is not combined with any other form. This cannot be simplified by using any law of exponents.

Similarly, $2^4 + 3^5 = 16 + 243 = 259$. Here again, we do not combine exponents while **adding** or **subtracting** powers of the same or different bases.

(iii) Laws of exponents are not applicable when base is 0.

Common Errors

(i) $2^5 + 2^3 = 2^{5+3}$

(ii) $(2^9)^3 = 2^{12}$

(iii) $(2^5)^3 = 2^{5^3}$

(iv) $0^0 = 0$

(v) $5^9 \div 5^4 = 5^{9/4}$

(vi) $5^8 \times 5^3 = (5+5)^{8+3} = 10^{11}$

(vii) $2^0 = 3^0 = 4^0$ therefore $2 = 3 = 4$

(viii) $4^3 \quad 5^3 = (20)^{3+3} = 20^6$

(ix) $3^4 + 4^4 = 7^4$

(x) Standard form of 325900000 is $32.59 \quad 10^7$

(xi) $2^3 = 6$

(xii) $3^{-1} = -3$

Through the following exercise you may evaluate learning of the concepts.

Exercise

1. Express 625 as a power of 5.

2. Express the following numbers as product of powers of prime factors.

   (i) 144 (ii) 1296 (iii) 10000

3. Find (i) $(\quad 1)^{10}$ (ii) $(-1)^{15}$ (iii) $(\quad 6)^4$

4. Compare : $(2^3)^3$ and $(3^2)^2$

5. Simplify :

6. Simplify : $(3^0 + 9^0) \times 4^0$

7. Simplify and write in exponential form

(i) $(\quad 2)^{-4} \times (2)^{-3}$

(ii) $3^{-2} \times 3^{4} \times 3^{-5}$

8. Find the value of $[(4)^3 \div (4)^8] \times 2^{-2}$

9. Find the value of $\left(\frac{1}{3}\right)^{-2} + \left(\frac{1}{2}\right)^{-2} + \left(\frac{1}{5}\right)^{-2}$

10. If $\frac{p}{q} = \left(\frac{3}{2}\right)^{-2} \div \left(\frac{6}{7}\right)^{0}$, find the value of $\left(\frac{p}{q}\right)^{-3}$

11. Find the value of m for which

$5^{m} \div 5^{-4} = 5^{2}$

12. Express the following numbers in standard form:

(i) 201,000,000,000,000

(ii) 149,600,000,000

(iii) 0.000,000,000,0054

(iv) 0.000,000,000,000,000,42

13. Write True or False for the following statements:

(i) $x^{m} \div x^{n} = x^{m-n}$ is true if $x > 0$ is an integer and $m \geq n$.

(ii) $x^{0} \times x^{0} = \frac{x^{0}}{x^{0}}$ is true for all non zero value of x.

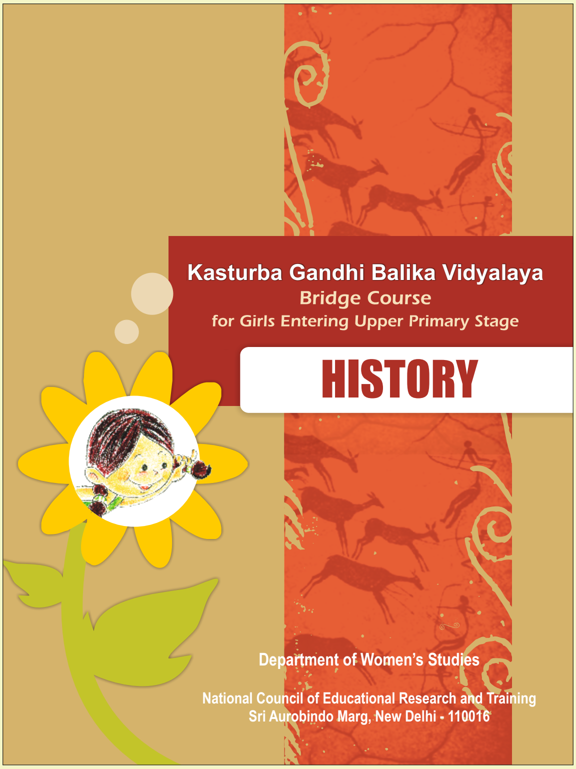
(iii) $(x^{-3})^{-4} = x^{12}$ for $x > 0$

(iv) $10 \times 10^{10} = 100^{10}$

(v) $2^{5} > 5^{2}$

(vi) $2^{2} \times 3^{3} = 6^{5}$

(vii) $2^{0} = (100)^{0}$



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