Proportional reasoning is important in scientific understanding and everyday life, but many children have difficulty mastering it. The present study investigated the effectiveness of a schematic representation in improving children's proportional reasoning. Ten-year-old students participated in three 30-minute sessions, the ‘diagram group’ solved proportion problems using a schematic diagram; the ‘non-diagram’ group solved the same problems without the diagram and the ‘unseen-control’ group participated only in the assessments. The groups did not differ at pre-test. On post-test proportion problems, the diagram-group performed significantly better than the unseen-control group whereas the non-diagram group did not. The data suggests that the schematic representation is an effective intermediate step before introducing the proportions formula. The salient steps involved in learning to use the diagram were also identified from children’s responses. These findings have curricular and pedagogical implications.

INTRODUCTION

Proportional reasoning constitutes an important mathematical understanding, widely applicable in professional and everyday life (Vergnaud, 1983). It is also an important element of the mathematics curriculum in later school years. However, proportional reasoning has been found to be difficult even for early adolescents. Only 60-70% of 12-13-year-old students use proportional reasoning with success in ratio-comparison and missing-value proportion problems (Noelting, 1980a, b; Vergnaud et al., 1979, Karplus et al., 1983). Thus, it is important to investigate which pedagogical strategies might be of value in facilitating children’s proportional reasoning.

Following Vergnaud’s (1983) classification, the present study focused on isomorphism of measures problems. These problems have a simple direct proportional relation between two measures, like the number of books purchased and the price paid for them. Vergnaud makes a distinction between the scalar and functional relations relevant in these problems. The relation between two values in the same measure space is the scalar relation whereas the relation between corresponding values of the two measure spaces is the functional relation. Based on this distinction, there can be two solution strategies for missing-value proportion problems. Scalar solution uses the scalar factor in finding the answer and functional solution uses the invariant functional relation in computing the solution (Nunes and Bryant, 1996), as shown in figure 1.

![Fig. 1: a) Scalar solution b) Functional solution](image-url)
Preference for scalar solutions

Children seem to show a clear preference for informal, building-up strategies which are essentially scalar in nature (Hart, 1981). Even when the computation required to implement a functional solution is simpler, children continue to adopt the scalar approach and make calculation errors (Vergnaud, 1979; Nunes et al, 1993). Saxe (1991) reports the persistence of the scalar preference in amount-price situations, which are familiar to children as involving co-variation between the two measures (Nunes and Bryant, 1996), and thus give some meaning to a functional solution.

The preference for scalar over functional strategies can be understood in terms of the meanings underpinning them. Scalar solutions relate to children’s informal building-up strategies like doubling and halving (Nunes and Bryant, 1996). These replication (or its inverse) strategies deeply rooted and are present from a very young age, even before the start of formal mathematics instruction (e.g. Piaget, 1965; Frydman and Bryant, 1988). This might explain why children continue to use correspondence reasoning to solve proportion problems even beyond the primary school (Nunes and Bryant, 2009). In contrast, reasoning functionally requires children to have an explicit understanding that there is a fixed multiplicative relationship between the two measures (Nunes and Bryant, 1996).

This preference, in itself, is not a matter of concern. However, children might get stuck and make mistakes in problems where it is difficult to find a scalar operator. At a deeper level, this preference might also mean that children have not grasped the logic of a functional relationship between variables (Nunes and Bryant, 1996), which is of essence in modelling. Modelling is defined as ‘the process of representing the world and operating on the representations to come to conclusions about the world’ (Nunes, 2012). The linear function is the simplest of these models (Ponte, 1992) and understanding this concept might serve as a basis for more complex functions in science and mathematics.

In their research synthesis, Nunes and Bryant (2009) conclude that a successful teaching strategy to address children’s difficulty with functional reasoning might be to offer them a way to represent both scalar and functional relations between quantities in order to link their informal knowledge to more formal methods.

Different forms of representations

A widely-taught symbolic representation for proportion problems is \( \frac{a}{b} = \frac{c}{x} \). This equation, called the rule-of-three, can be used to represent the quantities and relations given in the problem to solve for the fourth, missing value. This formula can be quite useful as it is suited to represent both scalar and functional relations. However, children do not adopt this formula and consistently fall back on their informal, scalar approach (Hart, 1981; Nunes et al., 1993; Vergnaud et al., 1979).

Researchers at the Freudenthal Institute emphatically assert that formalising using symbolic representations should be the last stage of instruction (Freudenthal, 1971; Gravemeijer, 1997; Streefland, 1985b in van den Heuvel-Panhuizen, 2003). Instead, a bottom-up approach should be used in which models built using manipulatives and graphical representations act as intermediate steps to support children’s progress from their informal knowledge to this final stage involving abstractions (Gravemeijer, 1997).
Bruner (1966, p.49) similarly argues that an optimum sequence of instructional material mirrors the sequence of intellectual development, from enactive through iconic to symbolic representations of the world. Thus, children should have ‘perceptual-manipulative insights’ and ‘a stock of visual images’ embodying the abstract mathematical knowledge before they are introduced to formal mathematical notation (p.62). He further claims that at a later stage, children can fall back on this imagery when the symbolic representations do not help clarify the problem. It has also been argued that using visual representation might help in the ‘systematic analysis and elaboration of the problem situation’ and hence increase the chances of solving mathematical problems correctly (De Bock et al., 2003, p. 445-446).

Use of schematic representations for solving proportion problems

Streefland and his colleagues (Van den Brink and Streefland, 1979; Streefland, 1984, 1985a) present a detailed instructional programme on proportions. The lessons involve a gradual progression from iconic representations to more schematic ones to facilitate the numerical processing of proportion. Figure 2 illustrates this progression. Streefland argued that this schematic diagram is efficient and context-independent and thus could act as a general schema applicable to similar situations involving ratio invariance.

![Fig. 2: Progression from iconic to more schematic representation (adapted from Streefland, 1984)](image)

This work is very ingenious and is well-situated in theory. However, the absence of appropriate controls and formal assessment of children’s knowledge make it difficult to comment on the effectiveness of this programme. A second limitation seems to be the sole focus on scalar transformations, with only a fleeting reference to the possibility of representing functional relations using the schematic diagram.

For simple proportion problems, Vergnaud (1994) recommends the use of the ‘table-and-arrow diagram’, an example of which was shown in figure 1. As can be seen, his representation is not very different from the one proposed by Streefland. Vergnaud considers this diagram to be a good pre-algebraic representation accessible to most children as it uses the two-dimensional space well to represent the relevant properties of the simple proportion. Mechmandarov (1987, in Nesher, 1992) and Sellke et al. (1991) investigated the effectiveness of the schematic diagram in understanding proportion problems. Their findings suggest that it has value in helping (sixth- and seventh-grade) students recognise the multiplicative relations in these problems and in arriving at the correct solution. However, these studies had methodological as well as conceptual limitations. Methodologically, it is not clear whether Mechmandarov (1987) identified an appropriate control group and whether there was random assignment to the teaching groups. From a conceptual perspective, it is a limitation that they focused only on proportion problems which always had ‘1’ as one of the given values. The problems without this unit measure may place very different demands on children because the unique properties of multiplication and division by 1 cannot be used in those problems. More importantly, these
studies did not use the diagram to explore functional relations. This too, perhaps, was an artefact of using unit-measure proportion problems.

In summary, there is a need for well-controlled empirical studies that investigate the effectiveness of schematic diagrams in helping children understand relations in proportion problems.

AN EMPIRICAL EVALUATION OF STREEFLAND’S SCHEMATIC DIAGRAM

In this brief intervention study, the hypothesis is that the use of the diagram proposed by Streefland is an effective way of supporting children’s understanding of proportionality. It is thus predicted that, if children learn to use this diagram, they can quickly progress in their ability to solve proportions problems, whereas children with a similar amount of practice will not show substantial gains if they do not use the diagram as a support during the learning phase. It is also hypothesized that the diagram effectively helps children identify proportions problems as different from additive problems. Thus, their performance on additive problems should neither improve nor decay as a consequence of practice in the use of Streefland’s diagram.

METHOD

Participants

The participants were 63 normally developing Year 5 children (29 boys, 34 girls) from four State schools in Oxfordshire, which serve a varied clientele in socio-economic terms. The mean age at pre-test was 10.16 years ($SD=31$); the range was 9.45 to 10.69 years. There were 21 participants in each of the three groups (described below). The differences between the groups in mean age at pre-test were not significant, $F(2, 60) = 0.046, p > 0.05$.

Design

A random assignment experimental design was used. All the children participated in a pre-test and a post-test. The participants were randomly assigned to one of the three groups. Between the pre- and the post-test, the diagram group received instruction on the use of schematic diagram to solve missing-value proportion problems. The non-diagram group spent the same amount of time working on the same problems but it was not taught the use of the schematic diagram. The unseen-control group participated only in the pre- and the post-tests. Because the study was a teaching experiment carried out in school settings, the ecological validity of the study is not seriously compromised.

Tests and Testing Procedure

The pre- and post-tests were paper-and-pencil tests. The problems consisted of varied contexts believed to be engaging and relevant for children. The pre-test, developed by Nunes et al. (2007), had 10 problems on additive reasoning and 4 on proportional reasoning. Some of these problems were found in a longitudinal study to predict children’s mathematical achievement in standardised tests 3 and 5 years later (Nunes et al., 2012). In the post-test, further questions were added, with proportion questions adapted from Kaput and West (1994) and Nunes and

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1 This project was part of a wider investigation and the pre-test included more test items, which were not analysed in the project.
Bryant (2012). In all, there were 15 additive and 12 proportion problems. The proportion problems were split into four groups—problems with easy scalar solutions, problems with easy functional solutions, problems which could be solved both ways, and finally, the proportion problems from the pre-test. These four groups of proportion problems were randomly interspersed with additive reasoning problems. These different problems types helped to check whether children had learned to discern the situations for which the schematic representation can be used and, if so, whether the representation offered them the flexibility to use both scalar and functional solutions.

The tests were administered in each school in a group session. The children received individual booklets to enter their answers. The problems were read out loud by the researcher; no reading was required from the children. Figures illustrating each problem were projected on a screen in front of the class so that the researcher could guide the children throughout the assessment. Children were asked to write each answer and show how they worked it out.

**Intervention**

The study was carried out over five weeks. The first week involved pre-testing in the schools, followed by three weekly intervention sessions and post-testing in the fifth week. In each intervention session, the researcher worked with a pair or trio of children for about thirty minutes in a quiet area within the school. Each question was read out while presenting illustrations on a laptop screen. The children were asked to try to solve the problems and discuss their solutions. They received worksheets for each problem and were allowed to use calculators. The researcher guided the children, directing their attention or asking questions which gave them clues whenever they got stuck. The meaning of the answers was regularly explored. All the groups were audio-recorded. Children were rewarded for their time and effort with stickers at the end of every session and a certificate at the end of the intervention.

All the problems used in the intervention were missing-value proportion problems with non-unit ratios. A variety of contexts was used, which were believed to be engaging for children of this age. Because scalar reasoning is easier for children, the first session contained problems designed to have an easy scalar solution. The second session was aimed at helping children to focus on the functional relations. Thus, the problems were designed to have an easier functional solution. The third session was aimed at improving children’s ability to flexibly move between scalar and functional approaches and, thus, problems were a mixed set of easier scalar or easier functional solutions. These were common features of the sessions for the two active groups. The unique pedagogical strategy for each group is described below.

**Intervention for the Diagram Group**

The first session was very important in terms of introducing the diagram to the children. In the first problem, the researcher presented the schematic representation as the linking table and modelled the use of the diagram as shown in figure 3. The problem was, “Emily bought 4 balloons and paid £10 for them. She went back and bought 12 more balloons for her class. How much did she have to pay for 12 balloons?”
The arrow was used in the illustration but the scalar relation \((x3)\) was provided by the children. If a group could not answer how many times 4 was 12, they were asked to find out the price for 8 balloons, as doubling is easier for children to understand. Once they understood how the diagram worked for ‘times 2’, children were able to follow the explanation for ‘times 3’. They were then asked to make the diagram on their worksheet. In rest of the session, the researcher guided the children’s attempts to use the diagram, progressively reducing the number of prompts given to them.

In the second session, the functional approach was demonstrated as the new way of using the linking table. It was used to convey visually the idea of a constant multiplicative relation between the two variables. For example, in the problem, “In a school, there is a hamster. It eats 12 scoops of food in 4 days. How much food will the hamster need for 7 days?” the researcher first marked the given pair (4 days and 12 scoops) on the respective lines. Since halving is easily understood by children, the researcher asked about the number of scoops needed for 2 days and then for 1 day. Following this, children’s attention was drawn to the constant vertical link \((x3)\) between the two lines, as illustrated in figure 4. They were told that they did not need to show the intermediate steps in subsequent problems even if they knew how to find out the functional ratio connecting the two measures directly.

In the third session, children revised solving one scalar and one functional problem using the diagram. After they solved these problems, they were shown a slide, as in figure 5, depicting both ways of using the diagram to emphasise the contrast.
For all subsequent questions, children were first asked to judge if a scalar or a functional solution would be easier and justify their choice before starting to solve the problem.

**Intervention for the Non-diagram Group**

The sessions for the non-diagram group were similar to those for the diagram-group except the use of the schematic diagram. To maintain this group’s motivation in working through the problems, children were guided to use counters for problem-solving. For instance, for the first problem, children were given counters to represent 12 balloons and were asked to see how many groups of 4 could be made with them. After they made three groups of four, the children were prompted to use another kind of counters to represent money. The children then built up the groups by placing 10 money counters with each group of 4 balloon counters. Finally, they added the number of money counters to arrive at the solution. If a group did not understand, they were first asked to find out for 8 balloons by making groups of four with 8 balloon counters.

From the second session, to maintain similarity in the amount of pedagogical support, the non-diagram group received written prompts to help them put down the information in a systematic manner. This representation, illustrated in figure 6, was similar to the rule-of-three and involved a more symbolic approach to the solution.

![Fig. 6: Symbolic representation for the hamster question](image)

In the third session, they were shown a slide with both the solutions to emphasise the contrast, as seen in figure 7. For all subsequent problems, children were asked to decide first whether it was easier to find the ratio by moving between the two values within a measure space or between the two measure spaces.

![Fig. 7: Contrast between functional and scalar solutions shown to the non-diagram group](image)

**RESULTS**

**Children’s responses to learning the diagram**

The children’s productions and analyses of their difficulties during the sessions were used to identify the steps that they need to take in order to use the diagram: (a) setting up the diagram; (b) finding the scalar or functional ratio; and (c) applying the ratio to find the solution.
Setting up the diagram involves drawing two lines for the measure spaces, marking the three quantities on the respective lines, with functionally related values in correspondence. In session 1, almost all children made mistakes in setting up the diagram in at least one question. With increased practice and visual prompts on the worksheets, the number of children making this mistake decreased substantially in the subsequent sessions.

The next step is to quantify the relation (scalar or functional) that connects a chosen pair of values. Most children did not realise that they could use division to find the scalar or functional relation: they searched their knowledge of multiplication tables and did not realise how they could use the calculators that had been provided.

After quantifying the relation between the quantities, the next step is to use it to find the missing value. Depending on the problem, this would involve a division or a multiplication, but because the children focused on the multiplication table, their preferred response was to multiply the relation by the third quantity.

The analysis of children’s responses thus revealed the procedural and conceptual challenges that children face when attempting to use the diagram.

Results from Quantitative Analyses

Pre-test differences

A one-way analysis of variance (ANOVA) for independent groups showed that the groups’ pre-test scores did not differ significantly, $F(2, 60) = 0.04, \quad p = .97$. Thus, if the groups are found to vary in their post-test performance, the results cannot be attributed to differences that existed before the intervention.

Effect of learning to use schematic diagram on proportional reasoning

The prediction was that the diagram group, who received instruction on the schematic diagram, would perform better on proportional reasoning problems than children in the unseen-control group at post-test, but the non-diagram group would not, as practice without the diagram would not suffice for significant progress in learning. To test this prediction, a one-way ANCOVA was carried out with the proportion scores at pre-test as the covariate. The group means for proportion problems at post-test, after adjusting for the effect of the covariate, are presented in table 1.

<table>
<thead>
<tr>
<th>Group</th>
<th>Mean</th>
<th>$SD$</th>
<th>Adjusted Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagram</td>
<td>7.71</td>
<td>3.33</td>
<td>7.71</td>
</tr>
<tr>
<td>Non-diagram</td>
<td>6.62</td>
<td>4.36</td>
<td>6.43</td>
</tr>
<tr>
<td>Unseen-control</td>
<td>5.71</td>
<td>3.82</td>
<td>5.90</td>
</tr>
</tbody>
</table>

Group membership did not have a significant effect on the post-test proportion scores, $F(2, 59) = 2.69, \quad p = .08$; but it indicated a trend. To increase statistical power of the comparisons (Keppel and Wickens, 2004), only two comparisons were made: the unseen-control group was compared with each of the intervention groups. With a stricter Sidak-corrected alpha value, the
planned contrasts revealed that the diagram group performed significantly better on the proportion problems at post-test than the unseen-control group, $p = .028$. An effect size of $d = 0.51$ was obtained for this difference. There was no significant difference between the performance of the non-diagram and the unseen-control group, $p = .51$. This suggests that the intervention offered to the diagram group was effective in improving children’s proportional reasoning when compared to no additional instruction.

**The effect of learning to use schematic representation on additive reasoning**

It was predicted that learning to use the diagram would have a specific effect on proportional reasoning and would have neither positive nor negative effect on additive reasoning. An ANCOVA carried out to test this prediction revealed that the effect of group membership on post-test additive scores was not significant after controlling for the additive scores at pre-test, $F (2, 59) = 0.53, p = .59$. Thus, the three groups performed at a similar level on the additive reasoning problems at post-test. This suggests that the diagram is specific for multiplicative reasoning problems.

**Association of using the schematic representation with performance on proportion problems**

Children may benefit from learning the diagram as a tool only if they use the tool. Also, because children prefer scalar solutions, the benefit of the tool may be more noticeable in problems in which a scalar solution is not easily implemented. To explore these ideas, the association of using the diagram at post-test with giving the correct response was tested for each proportion problem using Fisher’s *Exact* test.

As expected, for problems with an easy scalar solution, there was no significant association between using the diagram and arriving at the correct solution. A moderate but significant association was observed for three problems which did not have an easy scalar solution. However, for three other such problems, this association was not significant. Two of them involved difficult computation with decimals and one required application of the inverse ratio for the solution. These results hint at the potential role of the schematic representation in helping children solve problems that do not have an easy scalar solution, although this might not be so if the intermediate conceptual step or computation is difficult. An important observation was that the children were able to use the diagram flexibly for both scalar and functional problems.

**DISCUSSION**

The purpose of this study was to systematically investigate the effectiveness of a schematic representation for improving children’s proportional reasoning. The results extend earlier findings (Sellke et al., 1991; Mechmandarov, 1987) and suggest that practice on proportion problems using the schematic representation did improve children’s proportional reasoning in problems with non-unit ratios. The organization of the relevant elements of the problem on the diagram might have facilitated a systematic analysis and a planned solution (De Bock et al. (2003). The trend in the data also indicated that children might have found the diagram more accessible than a symbolic representation. In fact, diagram use was associated with success on some problems that did not have an easy scalar solution. These findings are significant in the context of the literature on children’s difficulty with proportional reasoning in general, and specifically with functional reasoning. The diagram used in this study seems to have helped children use their informal scalar approaches as well as explore functional relations.
Importantly, children did not apply the diagram to additive reasoning problems, even to ones with a missing-value format. Perhaps, this indicates that children realized that the diagram was useful for representing multiplicative relations in proportion problems.

This study also identified the salient procedural and conceptual steps involved in using the diagram. With explicit guidance, it was possible for children to learn to set up the diagram as a procedure. However, finding the ratio and applying it correctly caused greater difficulty in implementing the diagram. These decisions were conceptual, which were neither suggested nor supported by the diagram. This limitation of the diagram may be discussed in the context of Streefland’s (1985a) lesson series. The lessons started with visual representations that supported perceptual inferences, gradually moving to more mathematical representations. In the present study, children were straightaway presented with a schematic representation to solve proportion problems. This top-down approach may not have supported children’s reasoning at all the steps of the solution process. A longer intervention that allows for a more bottom-up approach might provide stronger evidence in favour of the schematic representation.

In spite of the limitations of a short-term intervention study, the findings are encouraging and have important pedagogical and curricular implications. In many countries, children are directly taught the rule-of-three formula to solve missing-value proportion problems, without adequate exploration of the relations involved in the problem situations (Nunes and Bryant, 2009). Research has shown that children frequently fail to adopt this formula, falling back on their informal scalar strategies instead. In India, the NCERT textbooks prescribe the use of unitary method to solve proportion problems. Though it seems that this method should make children aware of the functional relationship between two variables, children are more likely to think of scalar transformation in going from 1 unit to the desired number of units (Vergnaud, 1983). Given the importance of understanding functional relations, the present study suggests that the schematic diagram provides the children with an understandable intermediate step to explore both scalar and functional relations before moving on to a formal symbolic representation. The findings also indicate that children need instruction and guidance in learning to use the schematic representation to solve proportion problems successfully.
REFERENCES


